

INTRINSIC FLAT ARZELA-ASCOLI THEOREMS

CHRISTINA SORMANI

ABSTRACT. In this paper two Arzela-Ascoli Theorems are proven: one for uniformly Lipschitz functions whose domains are converging in the intrinsic flat sense, and one for sequences of uniformly local isometries between spaces which are converging in the intrinsic flat sense. A basic Bolzano-Weierstrass Theorem is proven for sequences of points in such sequences of spaces. In addition it is proven that when a sequence of manifolds has a pre-compact intrinsic flat limit then the metric completion of the limit is the Gromov-Hausdorff limit of regions within those manifolds. Open problems with suggested applications are provided throughout the paper.

1. INTRODUCTION

When studying sequences of Riemannian manifolds, one may use a variety of notions of convergence from $C^{k,\alpha}$ smooth convergence to Gromov-Hausdorff convergence as metric spaces. One needs to understand whether points and balls in the sequences converge to points and balls in limit spaces. So one proves Bolzano-Weierstrass theorems to produce converging subsequences of points. One needs to understand the limits of functions on these spaces and local isometries between these spaces. So one proves Arzela-Ascoli theorems for sequences of uniformly Lipschitz functions. Such theorems have been proven for Gromov-Hausdorff convergence by Gromov in [Gro99] and [Gro81] and by Grove-Petersen in [GP91]. They have been applied in these works as well as that of Cheeger-Colding [CC97], the author [Sor04], the author with Wei [SW01], Cheeger-Naber [CN11], and numerous other papers including Perelman's solution of the Poincare Conjecture [Per03]. In order to study open problems suggested by Gromov in [Gro12], one needs to prove similar theorems for intrinsic flat convergence.

Intrinsic flat convergence was introduced by Wenger and the author in [SW11] building upon work of Ambrosio-Kirchheim in [AK00]. It is defined for oriented Riemannian manifolds, M_j^m with boundary such that

$$(1) \quad \text{Vol}(M_j) \leq V_j \text{ and } \text{Vol}(\partial M_j) \leq A_j.$$

The limit spaces obtained under this convergence are called integral current spaces. They are either countably \mathcal{H}^m rectifiable metric spaces of the same dimension as the sequence or possibly the $\mathbf{0}$ space. Intrinsic flat limits may exist for sequences of manifolds with no Gromov-Hausdorff limit. However, if there is a Gromov-Hausdorff limit, $M_j \xrightarrow{\text{GH}} Y$, and one has uniform bounds on volume and boundary volume,

$$(2) \quad \text{Vol}(M_j) \leq V_0 \text{ and } \text{Vol}(\partial M_j) \leq A_0,$$

This material is based upon work supported by the National Science Foundation under Grant No. 0932078 000, while the author was serving as a visiting research professor at the *Mathematical Sciences Research Institute* in Berkeley, California, during the Fall of 2013. The author's research was also funded by an individual research grant NSF-DMS-1309360 and a PSC-CUNY Research Grant.

then a subsequence has an intrinsic flat limit, $M_{j_i} \xrightarrow{\mathcal{F}} X$ where $X \subset Y$ with the restricted distance, $d_X = d_Y$ [SW11]. It is possible that X is the $\mathbf{0}$ space or a strict subset of Y either because the sequence is collapsing or due to cancellation (see examples in [SW11]). This material is reviewed in Section 2.

This paper focuses on sequences of oriented Riemannian manifolds, M_j^m satisfying (1), or more generally integral current spaces, which converge in the intrinsic flat sense. The paper begins with the definition of converging and disappearing sequences of points [Definitions 3.1 and 3.2] and a proof that diameter is lower semicontinuous [Theorem 3.6]. Viewing balls within integral current spaces as integral current spaces themselves [Lemma 2.34] it is proven that, for almost every radius, balls around converging points have subsequences which converge to balls about their limit points [Lemma 4.1]. The necessity of taking a subsequence is shown in Example 4.3. If a sequence of points disappears, the balls of small radius about those points converge to the $\mathbf{0}$ space [Lemma 4.1]. Lemma 4.4 examines how the intrinsic flat distance may be estimated when the spaces are rescaled. Although technical, these lemmas are key steps in the subsequent theorems.

Theorem 5.1 states that if Riemannian manifolds M_i converge in the intrinsic flat sense to a nonzero precompact limit space, M , then there are open submanifolds $N_i \subset M_i$ such that $N_i \xrightarrow{\text{GH}} M$. This theorem and Remark 5.2 also describe the volumes of these submanifolds as well as what happens when M_i are integral current spaces. Section 5 also contains a few related open questions within remarks concerning possible extensions and applications of this theorem.

The first Arzela-Ascoli theorem, Theorem 6.1, states that if a sequence of functions, $F_i : M_i \rightarrow W$ where $M_i \xrightarrow{\mathcal{F}} M_\infty$ and W is compact and $\text{Lip}(F_i) \leq K$, then there is a converging subsequence $F_i \rightarrow F_\infty$ where $F_\infty : M_\infty \rightarrow W$ also has $\text{Lip}(F_\infty) \leq K$. A precise description as to exactly how $F_i \rightarrow F_\infty$ is given. Remarks 6.2 and 6.3 concern the impossibility and possibility of extending this theorem to allow the ranges to converge in the intrinsic flat and Gromov-Hausdorff sense respectively.

Theorem 7.1 is a Bolzano-Weierstrass type theorem for points $p_i \in M_i$ such that $M_i \xrightarrow{\mathcal{F}} M_\infty$. Since it is known that points may disappear in the limit [Remark 3.3], it is necessary to add a condition to obtain a subsequence with a limit point p_∞ . In Theorem 7.1, the extra condition is that for almost every sufficiently small radius there is a uniform bound on the intrinsic flat distance between the balls about p_i and $\mathbf{0}$. Remark 2.38 discusses how one can obtain such a bound when needed.

The second Arzela-Ascoli Theorem proven here in Theorem 8.1. Here the domains and ranges of the functions converge in the intrinsic flat sense and have uniform upper bounds as in (2). The functions are assumed to be local isometries which are isometries on balls of fixed radius. It is shown that a subsequence of the functions converges to a limit function which is also a local isometry. If the functions are surjective, then so is the limit. The case where the limit spaces are possibly the $\mathbf{0}$ space is also considered. Remark 8.4 discusses a possible extension of this theorem to uniformly locally bi-Lipschitz functions or more simply uniformly bi-Lipschitz functions. Remark 8.2 discusses the necessity of various conditions in Theorem 8.1.

In Section 9 an example is presenting showing how these theorems can be applied to prove certain sequences of Riemannian manifolds have no intrinsic flat limit. Additional applications to construct examples which do have specific limits will appear in joint work with Basilio [BS14].

Section 10 includes remarks describing the possible additional applications of the various theorems in this paper. In particular one may be able to apply Theorem 8.1 to answer a question posed by Gromov in [Gro12] concerning the intrinsic flat limits of tori whose universal covers have almost maximal volume growth in the sense described by Burago-Ivanov in [BI95]. See Remark 10.1. Additional possible applications of Theorem 8.1 to extend work of the author with Wei are described in Remarks 10.2 and 10.3. It may also be possible to apply Theorem 6.1 to study the limits of harmonic functions, eigenfunctions and heat kernels. See Remark 10.4. Finally in Remark 10.5, it is described how one may be able to apply Theorem 7.1 to prove that the intrinsic flat and Gromov-Hausdorff limits of Riemannian manifolds with uniform lower Ricci curvature bounds agree extending a theorem of the author with Wenger in [SW10].

The author would like to thank Jacobus Portegies and Raquel Perales for reading an earlier version of this paper and providing feedback. She would like to thank Blaine Lawson for suggesting that the basic properties of intrinsic flat convergence should appear in their own paper separate from the more technical theorems involving the Gromov Filling Volume which appear in [Sor13].

2. BACKGROUND

Here the key definitions and theorems applied in this paper are reviewed. Please keep in mind that this is by no means a complete introduction to Gromov-Hausdorff convergence and Intrinsic Flat convergence. We review only the notions that are applied in this paper. In fact, the primary reason for combining Theorems 5.1, 6.1, 7.1, and 8.1 together into this paper is because these four theorems can be proven using the same background material. Other related theorems appearing in [Sor13] all require additional results of Gromov and Ambrosio-Kirchheim.

Those who have already studied the notion of Intrinsic Flat convergence in the initial paper by the author with Wenger [SW11], should still review Subsections 2.1, 2.3 and 2.7 which cover material not presented there. Those who have never studied Gromov-Hausdorff or Intrinsic Flat convergence will find the entire background section useful as a very brief but self contained introduction to the subjects. As the author sees no reason to restate theorems, definitions and remarks, some of these statements have been repeated exactly as stated in prior background sections written by the author elsewhere.

2.1. A Review of Gromov-Hausdorff Convergence. Throughout this paper, Gromov's definition of an isometric embedding will be used:

Definition 2.1. *A map $\varphi : X \rightarrow Y$ between metric spaces, (X, d_X) and (Y, d_Y) , is an isometric embedding iff it is distance preserving:*

$$(3) \quad d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \quad \forall x_1, x_2 \in X.$$

Observe that this does not agree with the Riemannian notion of an isometric embedding.

The following is one of the more beautiful definitions of the Gromov-Hausdorff distance:

Definition 2.2 (Gromov). *The Gromov-Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as*

$$(4) \quad d_{GH}(X, Y) := \inf d_H^Z(\varphi(X), \psi(Y))$$

where Z is a complete metric space, and $\varphi : X \rightarrow Z$ and $\psi : Y \rightarrow Z$ are isometric embeddings and where the Hausdorff distance in Z is defined as

$$(5) \quad d_H^Z(A, B) = \inf\{\epsilon > 0 : A \subset T_\epsilon(B) \text{ and } B \subset T_\epsilon(A)\}.$$

Gromov proved that this is indeed a distance on compact metric spaces in the sense that $d_{GH}(X, Y) = 0$ iff there is an isometry between X and Y in [Gro99]. Gromov proved the following embedding theorem in [Gro81]:

Theorem 2.3 (Gromov). *If a sequence of compact metric spaces, X_j , converges in the Gromov-Hausdorff sense to a compact metric space X_∞ ,*

$$(6) \quad X_j \xrightarrow{GH} X_\infty$$

then in fact there is a compact metric space, Z , and isometric embeddings $\varphi_j : X_j \rightarrow Z$ for $j \in \{1, 2, \dots, \infty\}$ such that

$$(7) \quad d_H^Z(\varphi_j(X_j), \varphi_\infty(X_\infty)) \rightarrow 0.$$

This theorem allows one to define converging sequences of points:

Definition 2.4. *One says that $x_j \in X_j$ converges to $x_\infty \in X_\infty$, if there is a common space Z as in Theorem 2.3 such that $\varphi_j(x_j) \rightarrow \varphi_\infty(x)$ as points in Z . If one discusses the limits of multiple sequences of points then one uses a common Z and the same collection of φ_j to determine the convergence. This avoids difficulties arising from isometries in the limit space. Then one immediately has*

$$(8) \quad \lim_{j \rightarrow \infty} d_{X_j}(x_j, x'_j) = d_{X_\infty}(x_\infty, x'_\infty)$$

whenever $x_j \rightarrow x_\infty$ and $x'_j \rightarrow x'_\infty$ via a common Z .

One can apply Theorem 2.3 to see that for any $x_\infty \in X_\infty$ there exists $x_j \in X_j$ converging to x_∞ in this sense. Also observe that whenever x_j converges to x_∞ in this sense,

$$(9) \quad d_{GH}(B(x_j, r), B(x_\infty, r)) \leq d_H^Z(B(\varphi_j(x_j), r), B(\varphi_\infty(x_\infty), r)) \rightarrow 0 \quad \forall r > 0$$

if one views the balls $B(x_j, r) \subset X_j$ as metric spaces endowed with the restricted metric, d_{X_j} , from X_j . See the appendix of joint work of the author with Wei [SW04b] for a theorem concerning the induced length metrics. Theorem 2.3 also implies the following basic Bolzano-Weierstrass Theorem:

Theorem 2.5 (Gromov). *Given compact metric spaces, $X_j \xrightarrow{GH} X_\infty$, and $x_j \in X_j$ then a subsequence also denoted x_j converges to a point $x_\infty \in X_\infty$ in the sense described above.*

In particular, one sees that

$$(10) \quad X_j \rightarrow X_\infty \implies \lim_{j \rightarrow \infty} \text{Diam}(X_j) = \text{Diam}(X_\infty).$$

Gromov's embedding theorem can also be applied in combination with other extension theorems to obtain the following Gromov-Hausdorff Arzela-Ascoli Theorem. See also the appendix of a paper of Grove-Petersen [GP91] for a detailed proof and prior work of the author for a more general statement [Sor04].

Theorem 2.6 (Gromov). *Given compact metric spaces $X_j \xrightarrow{GH} X_\infty$ and $Y_j \rightarrow Y_\infty$ and equicontinuous functions $f_j : X_j \rightarrow Y_j$ in the sense that*

$$(11) \quad \forall \epsilon > 0 \exists \delta_\epsilon > 0 \text{ such that } d_{X_j}(x, x') < \delta \implies d_{Y_j}(f_j(x), f_j(x')) \leq \epsilon.$$

then there exists a subsequence, also denoted $f_j : X_j \rightarrow Y_j$ whuc converges to a continuous function $f_\infty : X_\infty \rightarrow Y_\infty$ in the sense that there exists common compact metric spaces Z, W , and isometric embeddings $\varphi_j : X_j \rightarrow Z, \psi_j : Y_j \rightarrow W$ such that

$$(12) \quad \lim_{j \rightarrow \infty} \psi_j(f_j(x_j)) = \psi_\infty(f_\infty(x_\infty)) \text{ whenever } \lim_{j \rightarrow \infty} \varphi_j(x_j) = \varphi_\infty(x_\infty).$$

Furthermore, if $\text{Lip}(f_j) \leq K$ then $\text{Lip}(f_\infty) \leq K$.

Observe in particular that if $y_j, y'_j \in Y_j$ converge to $y_\infty, y'_\infty \in Y_\infty$, where $Y_j \xrightarrow{\text{GH}} Y_\infty$, and $\gamma_j : [0, 1] \rightarrow Y_j$ are minimizing geodesics from y_j to y'_j , then a subsequence converges to $\gamma_\infty : [0, 1] \rightarrow Y_\infty$ which one can then show is a minimizing geodesic between x_∞ and x'_∞ . Thus geodesic metric spaces converge to geodesic metric spaces.

All these theorems are key ingredients in the many important works applying Gromov-Hausdorff convergence to better understand Riemannian Geometry. See the classic textbook of Burago-Burago-Ivanov [BBI01], the work of Cheeger-Colding [CC97] and the work of the author with Wei [SW01].

In this paper these theorems are extended, as far as possible, in the setting where one only has intrinsic flat convergence. Of course it is known that these theorems do not hold in their full strength in the setting where sequences of Riemannian manifolds are converging in the intrinsic flat sense. Examples in joint work of the author with Wenger in [SW11] demonstrate that (10) fails in general and that geodesics need not converge to geodesics. Nevertheless there are versions of these theorems which do hold.

2.2. Review of Ambrosio-Kirchheim Currents on Metric Spaces. In order to rigorously review the definition of the intrinsic flat distance, one needs a few key results of Ambrosio-Kirchheim. These results will also be applied later to prove our theorems.

In [AK00], Ambrosio-Kirchheim extend Federer-Fleming's notion of integral currents on Euclidean space to an arbitrary complete metric space, Z . In Federer-Fleming, currents were defined as linear functionals on differential forms [FF60]. This approach extends naturally to smooth manifolds but not to complete metric spaces which do not have differential forms. In the place of differential forms, Ambrosio-Kirchheim use DiGeorgi's $m + 1$ tuples, $\omega \in \mathcal{D}^m(Z)$,

$$(13) \quad \omega = f\pi = (f, \pi_1 \dots \pi_m) \in \mathcal{D}^m(Z)$$

where $f : X \rightarrow \mathbb{R}$ is a bounded Lipschitz function and $\pi_i : X \rightarrow \mathbb{R}$ are Lipschitz.

In [AK00] Definitions 2.1, 2.2, 2.6 and 3.1, an m dimensional current $T \in \mathbf{M}_m(Z)$ is defined. Here these are combined into a single definition:

Definition 2.7. *On a complete metric space, Z , an m dimensional **current**, denoted $T \in \mathbf{M}_m(Z)$, is a real valued multilinear functional on $\mathcal{D}^m(Z)$, with the following three required properties:*

i) Locality:

$$(14) \quad T(f, \pi_1, \dots, \pi_m) = 0 \text{ if } \exists i \in \{1, \dots, m\} \text{ s.t. } \pi_i \text{ is constant on a nbd of } \{f \neq 0\}.$$

ii) Continuity:

$$(15) \quad \text{Continuity of } T \text{ with respect to the ptwise convergence of } \pi_i \text{ such that } \text{Lip}(\pi_i) \leq 1.$$

iii) Finite mass:

$$(16) \quad \exists \text{ finite Borel } \mu \text{ s.t. } |T(f, \pi_1, \dots, \pi_m)| \leq \prod_{i=1}^m \text{Lip}(\pi_i) \int_Z |f| d\mu \quad \forall (f, \pi_1, \dots, \pi_m) \in \mathcal{D}^m(Z).$$

In [AK00] Definition 2.6 Ambrosio-Kirchheim introduce their mass measure:

Definition 2.8. *The mass measure $\|T\|$ of a current $T \in \mathbf{M}_m(Z)$, is the smallest Borel measure μ such that*

$$(17) \quad |T(f, \pi)| \leq \int_X |f| d\mu \quad \forall (f, \pi) \text{ where } \text{Lip}(\pi_i) \leq 1.$$

The mass of T is defined

$$(18) \quad M(T) = \|T\|(Z) = \int_Z d\|T\|.$$

In particular

$$(19) \quad |T(f, \pi_1, \dots, \pi_m)| \leq \mathbf{M}(T) |f|_\infty \text{Lip}(\pi_1) \cdots \text{Lip}(\pi_m).$$

Ambrosio-Kirchheim then define restrictions and push forwards:

Definition 2.9. [AK00][Defn 2.5] *The restriction $T \llcorner \omega \in \mathbf{M}_m(Z)$ of a current $T \in \mathbf{M}_{m+k}(Z)$ by a $k+1$ tuple $\omega = (g, \tau_1, \dots, \tau_k) \in \mathcal{D}^k(Z)$:*

$$(20) \quad (T \llcorner \omega)(f, \pi_1, \dots, \pi_m) := T(f \cdot g, \tau_1, \dots, \tau_k, \pi_1, \dots, \pi_m).$$

Given a Borel set, A ,

$$(21) \quad T \llcorner A := T \llcorner \omega$$

where $\omega = 1_A \in \mathcal{D}^0(Z)$ is the indicator function of the set. In this case,

$$(22) \quad \mathbf{M}(T \llcorner \omega) = \|T\|(A).$$

Definition 2.10. *Given a Lipschitz map $\varphi : Z \rightarrow Z'$, the push forward of a current $T \in \mathbf{M}_m(Z)$ to a current $\varphi_\# T \in \mathbf{M}_m(Z')$ is given in [AK00][Defn 2.4] by*

$$(23) \quad \varphi_\# T(f, \pi_1, \dots, \pi_m) := T(f \circ \varphi, \pi_1 \circ \varphi, \dots, \pi_m \circ \varphi).$$

Remark 2.11. *Observe that*

$$(24) \quad (\varphi_\# T) \llcorner (f, \pi_1, \dots, \pi_k) = \varphi_\# (T \llcorner (f \circ \varphi, \pi_1 \circ \varphi, \dots, \pi_k \circ \varphi))$$

and

$$(25) \quad (\varphi_\# T) \llcorner A = (\varphi_\# T) \llcorner (1_A) = \varphi_\# (T \llcorner (1_A \circ \varphi)) = \varphi_\# (T \llcorner \varphi^{-1}(A)).$$

In (2.4) [AK00], Ambrosio-Kirchheim show that

$$(26) \quad \|\varphi_\# T\| \leq [\text{Lip}(\varphi)]^m \varphi_\# \|T\|,$$

so that when φ is an isometric embedding

$$(27) \quad \|\varphi_\# T\| = \varphi_\# \|T\| \text{ and } \mathbf{M}(T) = \mathbf{M}(\varphi_\# T).$$

The simplest example of a current is:

Example 2.12. *If one has a bi-Lipschitz map, $\varphi : \mathbb{R}^m \rightarrow Z$, and a Lebesgue function $h \in L^1(A, \mathbb{Z})$ where $A \in \mathbb{R}^m$ is Borel, then $\varphi_\#[h] \in \mathbf{M}_m(Z)$ an m dimensional current in Z . Note that*

$$(28) \quad \varphi_\#[h](f, \pi_1, \dots, \pi_m) = \int_{A \subset \mathbb{R}^m} (h \circ \varphi)(f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

where $d(\pi_i \circ \varphi)$ is well defined almost everywhere by Rademacher's Theorem. Here the mass measure is

$$(29) \quad \|[\varphi_\#[h]]\| = h d\mathcal{L}_m$$

and the mass is

$$(30) \quad \mathbf{M}([h]) = \int_A h d\mathcal{L}_m.$$

In [AK00][Theorem 4.6] Ambrosio-Kirchheim define the following set associated with any integer rectifiable current:

Definition 2.13. *The (canonical) set of a current, T , is the collection of points in Z with positive lower density:*

$$(31) \quad \text{set}(T) = \{p \in Z : \Theta_{*m}(\|T\|, p) > 0\},$$

where the definition of lower density is

$$(32) \quad \Theta_{*m}(\mu, p) = \liminf_{r \rightarrow 0} \frac{\mu(B_p(r))}{\omega_m r^m}.$$

In [AK00] Definition 4.2 and Theorems 4.5-4.6, an integer rectifiable current is defined using the Hausdorff measure, \mathcal{H}^m :

Definition 2.14. *Let $m \geq 1$. A current, $T \in \mathcal{D}_m(Z)$, is rectifiable if $\text{set}(T)$ is countably \mathcal{H}^m rectifiable and if $\|T\|(A) = 0$ for any set A such that $\mathcal{H}^k(A) = 0$. One writes $T \in \mathcal{R}_m(Z)$.*

One says $T \in \mathcal{R}_m(Z)$ is integer rectifiable, denoted $T \in \mathcal{I}_m(Z)$, if for any $\varphi \in \text{Lip}(Z, \mathbb{R}^m)$ and any open set $A \in Z$, then

$$(33) \quad \exists \theta \in \mathcal{L}^1(\mathbb{R}^k, Z) \text{ s.t. } \varphi_{\#}(T \llcorner A) = [\theta].$$

In fact, $T \in \mathcal{I}_m(Z)$ iff it has a parametrization. A parametrization $(\{\varphi_i\}, \{\theta_i\})$ of an integer rectifiable current $T \in \mathcal{I}^m(Z)$ is a collection of bi-Lipschitz maps $\varphi_i : A_i \rightarrow Z$ with $A_i \subset \mathbb{R}^m$ precompact Borel measurable and with pairwise disjoint images and weight functions $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$(34) \quad T = \sum_{i=1}^{\infty} \varphi_{i\#}[\theta_i] \quad \text{and} \quad \mathbf{M}(T) = \sum_{i=1}^{\infty} \mathbf{M}(\varphi_{i\#}[\theta_i]).$$

A 0 dimensional rectifiable current is defined by the existence of countably many distinct points, $\{x_i\} \in Z$, weights $\theta_i \in \mathbb{R}^+$ and orientation, $\sigma_i \in \{-1, +1\}$ such that

$$(35) \quad T(f) = \sum_h \sigma_i \theta_i f(x_i) \quad \forall f \in \mathcal{B}^{\infty}(Z).$$

where $\mathcal{B}^{\infty}(Z)$ is the class of bounded Borel functions on Z and where

$$(36) \quad \mathbf{M}(T) = \sum_h \theta_i < \infty$$

If T is integer rectifiable $\theta_i \in \mathbb{Z}^+$, so the sum must be finite.

In particular, the mass measure of $T \in \mathcal{I}_m(Z)$ satisfies

$$(37) \quad \|T\| = \sum_{i=1}^{\infty} \|\varphi_{i\#}[\theta_i]\|.$$

Theorems 4.3 and 8.8 of [AK00] provide necessary and sufficient criteria for determining when a current is integer rectifiable.

Note that the current in Example 2.12 is an integer rectifiable current.

Example 2.15. *If one has a Riemannian manifold, M^m , and a bi-Lipschitz map $\varphi : M^m \rightarrow Z$, then $T = \varphi_{\#}[1_M]$ is an integer rectifiable current of dimension m in Z . If φ is an isometric embedding, and $Z = M$ then $\mathbf{M}(T) = \text{Vol}(M^m)$. Note further that $\text{set}(T) = \varphi(M)$.*

Definition 2.16. [AK00][Defn 2.3] The boundary of $T \in \mathbf{M}_m(Z)$ is defined

$$(38) \quad \partial T(f, \pi_1, \dots, \pi_{m-1}) := T(1, f, \pi_1, \dots, \pi_{m-1}) \in M_{m-1}(Z)$$

When $m = 0$, set $\partial T = 0$.

Note that $\varphi_{\#}(\partial T) = \partial(\varphi_{\#}T)$.

Definition 2.17. [AK00][Defn 3.4 and 4.2] An integer rectifiable current $T \in \mathbf{I}_m(Z)$ is called an integral current, denoted $T \in \mathbf{I}_m(Z)$, if ∂T defined as

$$(39) \quad \partial T(f, \pi_1, \dots, \pi_{m-1}) := T(1, f, \pi_1, \dots, \pi_{m-1})$$

has finite mass. The total mass of an integral current is

$$(40) \quad \mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T).$$

Observe that $\partial \partial T = 0$. In [AK00] Theorem 8.6, Ambrosio-Kirchheim prove that

$$(41) \quad \partial : \mathbf{I}_m(Z) \rightarrow \mathbf{I}_{m-1}(Z)$$

whenever $m \geq 1$. By (26) one can see that if $\varphi : Z_1 \rightarrow Z_2$ is Lipschitz, then

$$(42) \quad \varphi_{\#} : \mathbf{I}_m(Z_1) \rightarrow \mathbf{I}_m(Z_2).$$

2.3. Ambrosio-Kirchheim Slicing Theorem. As in Federer-Fleming, Ambrosio-Kirchheim consider the slices of currents:

Theorem 2.18. [Ambrosio-Kirchheim] [AK00][Theorems 5.6-5.7] Let Z be a complete metric space, $T \in \mathbf{I}_m Z$ and $f : Z \rightarrow \mathbb{R}$ a Lipschitz function. Let

$$(43) \quad \langle T, f, s \rangle := \partial(T \llcorner f^{-1}(-\infty, s]) - (\partial T) \llcorner f^{-1}(-\infty, s]),$$

so that

$$(44) \quad \partial \langle T, f, s \rangle = \langle -\partial T, f, s \rangle$$

and $\langle T_1 + T_2, f, s \rangle = \langle T_1, f, s \rangle + \langle T_2, f, s \rangle$. Then for almost every slice $s \in \mathbb{R}$, $\langle T, f, s \rangle$ is an integral current and one can integrate the masses to obtain:

$$(45) \quad \int_{s \in \mathbb{R}} \mathbf{M}(\langle T, f, s \rangle) ds = \mathbf{M}(T \llcorner df) \leq \text{Lip}(f) \mathbf{M}(T)$$

where

$$(46) \quad (T \llcorner df)(h, \pi_1, \dots, \pi_{m-1}) = T(h, f, \pi_1, \dots, \pi_{m-1}).$$

In particular, for almost every $s > 0$ one has

$$(47) \quad T \llcorner f^{-1}(-\infty, s] \in \mathbf{I}_{m-1}(Z).$$

Remark 2.19. Observe that for any $T \in \mathbf{I}_m(Z')$, and any Lipschitz functions, $\varphi : Z \rightarrow Z'$ and $f : Z' \rightarrow \mathbb{R}$ and any $s > 0$, one has

$$(48) \quad \langle \varphi_{\#}T, f, s \rangle = \varphi_{\#} \langle T, (f \circ \varphi), s \rangle.$$

2.4. Review of Convergence of Currents. Ambrosio Kirchheim's Compactness Theorem, which extends Federer-Fleming's Flat Norm Compactness Theorem, is stated in terms of weak convergence of currents. Definition 3.6 of [AK00] extends Federer-Fleming's notion of weak convergence (except that they do not require compact support):

Definition 2.20. *A sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ is said to converge weakly to a current T iff the pointwise limits satisfy*

$$(49) \quad \lim_{j \rightarrow \infty} T_j(f, \pi_1, \dots, \pi_m) = T(f, \pi_1, \dots, \pi_m)$$

for all bounded Lipschitz $f : Z \rightarrow \mathbb{R}$ and Lipschitz $\pi_i : Z \rightarrow \mathbb{R}$. One writes

$$(50) \quad T_j \rightarrow T$$

One sees immediately that $T_j \rightarrow T$ implies

$$(51) \quad \partial T_j \rightarrow \partial T,$$

and

$$(52) \quad \varphi_{\#} T_j \rightarrow \varphi_{\#} T.$$

However $T_j \llcorner A$ need not converge weakly to $T \llcorner A$ as seen in the following example:

Example 2.21. *Let $Z = \mathbb{R}^2$ with the Euclidean metric. Let $\varphi_j : [0, 1] \rightarrow Z$ be $\varphi_j(t) = (1/j, t)$ and $\varphi_{\infty}(t) = (0, t)$. Let $S \in \mathbf{I}_1([0, 1])$ be*

$$(53) \quad S(f, \pi_1) = \int_0^1 f d\pi_1$$

Let $T_j \in \mathbf{I}_1(Z)$ be defined $T_j = \varphi_{j\#}(S)$. Then $T_j \rightarrow T_{\infty}$. Taking $A = [0, 1] \times (0, 1)$, one has $T_j \llcorner A = T_j$ but $T_{\infty} \llcorner A = 0$.

Immediately below the definition of weak convergence [AK00] Defn 3.6, Ambrosio-Kirchheim prove the lower semicontinuity of mass: *If T_j converges weakly to T , then*

$$(54) \quad \liminf_{j \rightarrow \infty} \mathbf{M}(T_j) \geq \mathbf{M}(T).$$

and for any set, $A \subset Z$,

$$(55) \quad \liminf_{j \rightarrow \infty} \|T_j\|(A) \geq \|T\|(A).$$

Theorem 2.22 (Ambrosio-Kirchheim Compactness). *Given any complete metric space Z , a compact set $K \subset Z$ and $A_0, V_0 > 0$. Given any sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ satisfying*

$$(56) \quad \mathbf{M}(T_j) \leq V_0, \mathbf{M}(\partial T_j) \leq A_0 \text{ and } \text{set}(T_j) \subset K,$$

there exists a subsequence, T_{j_i} , and a limit current $T \in \mathbf{I}_m(Z)$ such that T_{j_i} converges weakly to T .

2.5. Review of Integral Current Spaces. The notion of an integral current space was introduced by the author and Stefan Wenger in [SW11]:

Definition 2.23. *An m dimensional metric space (X, d, T) is called an integral current space if it has a integral current structure $T \in \mathbf{I}_m(\bar{X})$ where \bar{X} is the metric completion of X and $\text{set}(T) = X$.*

Note that $\text{set}(\partial T) \subset \bar{X}$. The boundary of (X, d, T) is then the integral current space:

$$(57) \quad \partial(X, d_X, T) := (\text{set}(\partial T), d_{\bar{X}}, \partial T).$$

If $\partial T = 0$ then one says (X, d, T) is an integral current without boundary.

Remark 2.24. *Note that any m dimensional integral current space is countably \mathcal{H}^m rectifiable with orientated charts, φ_i and weights θ_i provided as in (34). A 0 dimensional integral current space is a finite collection of points with orientations, σ_i and weights θ_i provided as in (35). If this space is the boundary of a 1 dimensional integral current space, then as in Remark ??, the sum of the signed weights is 0.*

Example 2.25. *A compact oriented Riemannian manifold with boundary, M^m , is an integral current space, where $X = M^m$, d is the standard metric on M and T is integration over M . In this case $\mathbf{M}(M) = \text{Vol}(M)$ and ∂M is the boundary manifold. When M has no boundary, $\partial M = 0$.*

Definition 2.26. *The space of $m \geq 0$ dimensional integral current spaces, \mathcal{M}^m , consists of all metric spaces which are integral current spaces with currents of dimension m as in Definition 2.23 as well as the $\mathbf{0}$ spaces. Then $\partial : \mathcal{M}^{m+1} \rightarrow \mathcal{M}^m$.*

2.6. Review of the Intrinsic Flat Convergence. Recall that the flat distance between m dimensional integral currents $S, T \in \mathbf{I}_m(Z)$ is given by

$$(58) \quad d_F^Z(S, T) := \inf\{\mathbf{M}(U) + \mathbf{M}(V) : S - T = U + \partial V\}$$

where $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$. This notion of a flat distance was first introduced by Whitney in [Whi57] for chains and later adapted to rectifiable currents by Federer-Fleming [FF60]. The flat distance between Ambrosio-Kirchheim's integral currents was studied by Wenger in [Wen07]. In particular, Wenger proved that if $T_j \in \mathbf{I}_m(Z)$ has $\mathbf{M}(T_j) \leq V_0$ and $\mathbf{M}(\partial T_j) \leq A_0$ then

$$(59) \quad T_j \rightarrow T \text{ iff } d_F^Z(T_j, T) \rightarrow 0$$

exactly as in Federer-Fleming.

The intrinsic flat distance between integral current spaces was first defined in [SW11][Defn 1.1]:

Definition 2.27. *For $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2) \in \mathcal{M}^m$ let the intrinsic flat distance be defined:*

$$(60) \quad d_{\mathcal{F}}(M_1, M_2) := \inf d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2),$$

where the infimum is taken over all complete metric spaces (Z, d) and isometric embeddings $\varphi_1 : (\bar{X}_1, d_1) \rightarrow (Z, d)$ and $\varphi_2 : (\bar{X}_2, d_2) \rightarrow (Z, d)$ and the flat norm d_F^Z is taken in Z . Here \bar{X}_i denotes the metric completion of X_i and d_i is the extension of d_i on \bar{X}_i , while $\phi_{\#}T$ denotes the push forward of T .

In [SW11], it is observed that

$$(61) \quad d_{\mathcal{F}}(M_1, M_2) \leq d_{\mathcal{F}}(M_1, 0) + d_{\mathcal{F}}(0, M_2) \leq \mathbf{M}(M_1) + \mathbf{M}(M_2).$$

There it is also proven that $d_{\mathcal{F}}$ satisfies the triangle inequality [SW11][Thm 3.2] and is a distance [SW11][Thm 3.27] on the class of precompact integral current spaces up to current preserving isometry. In particular it is a distance on the class of oriented compact manifolds with boundary of a given dimension.

In [SW11] Theorem 3.23 it is also proven that

Theorem 2.28. [SW11][Thm 4.23] *Given a pair of precompact integral current spaces, $M_1^m = (X_1, d_1, T_1)$ and $M_2^m = (X_2, d_2, T_2)$, there exists a compact metric space, (Z, d_Z) , integral currents $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$, and isometric embeddings $\varphi_1 : \bar{X}_1 \rightarrow Z$ and $\varphi_2 : \bar{X}_2 \rightarrow Z$ with*

$$(62) \quad \varphi_{\#} T_1 - \varphi'_{\#} T_2 = U + \partial V$$

such that

$$(63) \quad d_{\mathcal{F}}(M_1, M_2) = \mathbf{M}(U) + \mathbf{M}(V).$$

Remark 2.29. *The metric space Z in Theorem 2.28 has*

$$(64) \quad \text{Diam}(Z) \leq 3 \text{Diam}(X_1) + 3 \text{Diam}(X_2).$$

This is seen by consulting the proof of Theorem 3.23 in [SW11], where Z is constructed as the injective envelope of the Gromov-Hausdorff limit of a sequence of spaces Z_n with this same diameter bound.

The following theorem in [SW11] is an immediate consequence of Gromov and Ambrosio-Kirchheim's Compactness Theorems:

Theorem 2.30. *Given a sequence of precompact m dimensional integral current spaces $M_j = (X_j, d_j, T_j)$ such that*

$$(65) \quad (\bar{X}_j, d_j) \xrightarrow{GH} (Y, d_Y), \quad \mathbf{M}(M_j) \leq V_0 \quad \text{and} \quad \mathbf{M}(\partial M_j) \leq A_0$$

then a subsequence converges in the intrinsic flat sense

$$(66) \quad (X_{j_i}, d_{j_i}, T_{j_i}) \xrightarrow{\mathcal{F}} (X, d_X, T)$$

where either (X, d_X, T) is the $\mathbf{0}$ current space or (X, d_X, T) is an m dimensional integral current space with $X \subset Y$ with the restricted metric $d_X = d_Y$.

Immediately one notes that if Y has Hausdorff dimension less than m , then $(X, d, T) = \mathbf{0}$. There are many examples of sequences of Riemannian manifolds which have no Gromov-Hausdorff limit but have an intrinsic flat limit. The first is Ilmanen's Example of an increasingly hairy three sphere with positive scalar curvature described in [SW11] Example A.7.

The following three theorems are proven in work of the author with Wenger [SW11]. These theorems with the work of Ambrosio-Kirchheim reviewed are key ingredients in the proofs of the theorems in this paper.

Theorem 2.31. [SW11][Thm 4.2] *If a sequence of integral current spaces has*

$$(67) \quad M_j = (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} M_0 = (X_0, d_0, T_0),$$

then there is a separable complete metric space, Z , and isometric embeddings $\varphi_j : X_j \rightarrow Z$ such that

$$(68) \quad d_F^Z(\varphi_{\#}T_j, \varphi_{0\#}T_0) \rightarrow 0$$

and thus $\varphi_{\#}T_j$ converges weakly to $\varphi_{0\#}T_0$ as well.

Theorem 2.32. [SW11][Thm 4.3] *If a sequence of integral current spaces has*

$$(69) \quad M_j = (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} \mathbf{0}$$

then one may choose points $x_j \in X_j$ and a separable complete metric space, Z , and isometric embeddings $\varphi_j : X_j \rightarrow Z$ such that $\varphi_j(x_j) = z_0 \in Z$ and

$$(70) \quad d_F^Z(\varphi_{\#}T_j, 0) \rightarrow 0$$

and thus $\varphi_{\#}T_j$ converges weakly to 0 in Z as well.

Theorems 2.31 and 2.32 combined with Ambrosio-Kirchheim's lower semicontinuity of mass [c.f. Remark 2.33] imply the following:

Theorem 2.33. *If a sequence of integral current spaces M_j converges in the intrinsic flat sense to an integral current space, M_∞ , then*

$$(71) \quad \liminf_{i \rightarrow \infty} \mathbf{M}(M_i) \geq \mathbf{M}(M_\infty)$$

Note that Theorems 2.31, 2.32 and 2.33 do not require uniform bounds on the masses or volumes of the M_j and ∂M_j .

2.7. Balls in Integral Current Spaces. Many theorems in Riemannian geometry involve balls,

$$(72) \quad B(p, r) = \{x \in X : d_X(x, p) < r\} \quad \bar{B}(p, r) = \{x \in X : d_X(x, p) \leq r\}.$$

Here a few basic lemmas are proven about balls in integral current spaces. These lemmas are new but so basic that they are best placed in this background section.

Lemma 2.34. *A ball in an integral current space, $M = (X, d, T)$, with the current restricted from the current structure of the Riemannian manifold is an integral current space itself,*

$$(73) \quad S(p, r) = (\text{set}(T \llcorner B(p, r)), d, T \llcorner B(p, r))$$

for almost every $r > 0$. Furthermore,

$$(74) \quad B(p, r) \subset \text{set}(S(p, r)) \subset \bar{B}(p, r) \subset X.$$

Proof. First one shows that $S(p, r) = T \llcorner B(p, r)$ is an integer rectifiable current. Let $\rho_p : \bar{X} \rightarrow \mathbb{R}$ be the distance function from p . Then by Ambrosio-Kirchheim's Slicing Theorem,

$$(75) \quad \partial(T \llcorner B(p, r)) = \partial(T \llcorner \rho_p^{-1}(-\infty, r))$$

$$(76) \quad = \langle T, \rho_p, r \rangle + (\partial T) \llcorner \rho_p^{-1}(-\infty, r)$$

$$(77) \quad = \langle T, \rho_p, r \rangle + (\partial T) \llcorner B(p, r)$$

where the mass of the slice $\langle T, \rho_p, r \rangle$ is bounded for almost every r . Thus

$$(78) \quad \mathbf{M}(\partial(T \llcorner B(p, r))) \leq \mathbf{M}(\langle T, \rho_p, r \rangle) + \mathbf{M}((\partial T) \llcorner B(p, r))$$

$$(79) \quad \leq \mathbf{M}(\langle T, \rho_p, r \rangle) + \mathbf{M}((\partial T) \llcorner X) < \infty.$$

So $S(p, r)$ is an integral current in \bar{X} for almost every r .

Next one proves (74). Recall that $x \in \text{set}(S(p, r)) \subset \bar{X}$ iff

$$(80) \quad 0 < \liminf_{s \rightarrow 0} \frac{\|S(p, r)\|(B(x, s))}{\omega_m s^m}$$

$$(81) \quad = \liminf_{s \rightarrow 0} \frac{\|T\|(B(p, r) \cap B(x, s))}{\omega_m s^m}$$

If $x \in B(p, r) \subset X$, then eventually $B(x, s) \subset B(p, r)$ and the liminf is just the lower density of T at x . Since $x \in X = \text{set}(T)$, this lower density is positive. If $x \in \bar{X} \setminus X$, then the liminf is 0 because it is smaller than the density of T at x , which is 0. If $x \notin \bar{B}(p, r)$, then the liminf is 0 because eventually the balls do not intersect. \square

One may imagine that it is possible that a ball is cusp shaped and that some points in the closure of the ball that lie in X do not lie in the set of $S(p, r)$. In a manifold, the set of $S(p, r)$ is a closed ball:

Lemma 2.35. *When M is a Riemannian manifold with boundary*

$$(82) \quad S(p, r) = (\bar{B}(p, r), d, T \llcorner B(p, r))$$

is an integral current space for all $r > 0$.

Proof. In this case,

$$(83) \quad \partial(T \llcorner B(p, r))(f, \pi_1, \dots, \pi_m) = (T \llcorner B(p, r))(1, f, \pi_1, \dots, \pi_m)$$

$$(84) \quad = T(\chi_{B(p, r)}, f, \pi_1, \dots, \pi_m)$$

$$(85) \quad = \int_M \chi_{B(p, r)} df \wedge d\pi_1 \wedge \dots \wedge d\pi_m$$

$$(86) \quad = \int_{B(p, r)} df \wedge d\pi_1 \wedge \dots \wedge d\pi_m$$

$$(87) \quad = \int_{B(p, r)} df \wedge d\pi_1 \wedge \dots \wedge d\pi_m$$

$$(88) \quad = \int_{\partial B(p, r)} f d\pi_1 \wedge \dots \wedge d\pi_m$$

So $\mathbf{M}(\partial(T \llcorner B(p, r))) = \text{Vol}_{m-1}(\partial B_p(r)) < \infty$.

Observe that $\bar{B}(p, r) \subset M$ is $\text{set}(S(p, r))$, by (80). If $d(x, p) = r$, then let $\gamma : [0, r] \rightarrow M$ be a curve parametrized by arclength running minimally from x to p . Then

$$(89) \quad B(\gamma(s/2), s/2) \subset B(x, s) \cap B(p, r).$$

and

$$(90) \quad \liminf_{s \rightarrow 0} \frac{\|S(p, r)\|(B(x, s))}{\omega_m s^m} = \liminf_{s \rightarrow 0} \frac{\|T\|(B(p, r) \cap B(x, s))}{\omega_m s^m}$$

$$(91) \quad \geq \liminf_{s \rightarrow 0} \frac{\|T\|(B(\gamma(s/2), s/2))}{\omega_m s^m}$$

$$(92) \quad \geq \liminf_{s \rightarrow 0} \frac{\text{Vol}(B(\gamma(s/2), s/2))}{2^m \omega_m (s/2)^m} \geq \frac{1/2}{2^m}$$

because in a manifold with boundary, the balls eventually lie within a half plane chart where all tiny balls are either uniformly close to a Euclidean ball or half a Euclidean ball. \square

Example 2.36. *There exist integral current spaces with balls that are not integral current spaces.*

Proof. Suppose one defines an integral current space, (X, d, T) where $X = S^2$ with the following generalized metric

$$(93) \quad g = dr^2 + (\cos(r)/r^2)^2 d\theta^2 \quad r \in [-\pi/2, \pi/2].$$

The metric is defined as

$$(94) \quad d(p_1, p_2) = \inf\{L_g(\gamma) : \gamma(0) = p_1, \gamma(1) = p_2\}$$

where

$$(95) \quad L_g(\gamma) = \int_0^1 g(\gamma'(t), \gamma'(t))^{1/2} dt$$

as in a Riemannian manifold. In fact this metric space consists of two open isometric Riemannian manifolds diffeomorphic to disks whose metric completions are glued together along corresponding points. The current structure T is defined by

$$(96) \quad T(f, \pi_1, \dots, \pi_m) = \int_{-\pi/2}^{\pi/2} \int_{S^1} f d\pi_1 \wedge \dots \wedge d\pi_m$$

$$(97) \quad = \int_{-\pi/2}^0 \int_{S^1} f d\pi_1 \wedge \dots \wedge d\pi_m + \int_0^{\pi/2} \int_{S^1} f d\pi_1 \wedge \dots \wedge d\pi_m$$

so that

$$(98) \quad \mathbf{M}(T) = \text{Vol}_m(r^{-1}[-\pi/2, 0)) + \text{Vol}_m(r^{-1}(0, \pi/2]) < \infty$$

and $\partial T = 0$

Setting p such that $r(p) = -\pi/2$, then $S(p, \pi/2)$ is a rectifiable current but its boundary does not have finite mass. This can be seen by taking q such that $(r(q), \theta(q)) = (0, 0)$, setting $\pi_1 = \rho_q$ and $f = \rho_p = r + \pi/2$ and observing that

$$(99) \quad |\partial(S(p, \pi/2))(f, \pi_1)| = |S(p, \pi/2)(1, f, \pi_1)|$$

$$(100) \quad = \left| \int_{B(p, \pi/2)} df \wedge d\pi_1 \right|$$

$$(101) \quad \geq \left| \int_{B(p, \pi/2 - \delta)} df \wedge d\pi_1 \right|$$

$$(102) \quad = \left| \int_{\partial B(p, \pi/2 - \delta)} f d\pi_1 \right|$$

$$(103) \quad = \left| \int_{\theta=-\pi}^{\pi} (\pi/2 - \delta) \frac{d\pi_1}{d\theta} d\theta \right|$$

$$(104) \quad = \left| \int_{\theta=-\pi}^{\pi} (\pi/2 - \delta) \frac{\cos(r)}{r^2} d\theta \right|$$

$$(105) \quad \geq 2\pi(\pi/2 - \delta) \frac{\cos(-\delta)}{\delta^2}$$

which is unbounded as δ decreases to 0. \square

Remark 2.37. Note that the outside of the ball, $(M \setminus B(p, r), d, T - S(p, r))$, is also an integral current space for almost every $r > 0$.

Remark 2.38. In some of the theorems in this paper, it will be important to estimate $d_{\mathcal{F}}(S(p, r), \mathbf{0})$. There are various ways to estimate this value. First observe that

$$(106) \quad d_{\mathcal{F}}(S(p, r), \mathbf{0}) \leq \min\{\mathbf{M}(S(p, r)), \mathbf{M}(\partial(S(p, r)))\}.$$

In addition, if one finds a comparison integral current space, N , such that

$$(107) \quad d_{\mathcal{F}}(S(p, r), N) < d_{\mathcal{F}}(N, \mathbf{0})/2$$

then by the triangle inequality

$$(108) \quad d_{\mathcal{F}}(S(p, r), \mathbf{0}) > d_{\mathcal{F}}(N, \mathbf{0})/2.$$

Recall that in joint work with Wenger [SW11], in joint work with Lakzian [LS13], and in joint work with Lee [LS14] various means of estimating the intrinsic flat distance are provided.

3. CONVERGING POINTS AND DIAMETERS

In this section the limits of points in sequences of integral current spaces that converge in the intrinsic flat sense are examined. See Definitions 3.1 and 3.2 and Lemma 3.4. The diameter is then proven to be lower semicontinuous. See Definition 3.5 and Theorem 3.6.

Before beginning, recall that Theorem 2.31 which was proven in work of the author with Wenger in [SW11] states that a sequence of manifolds which converges in the intrinsic flat sense can be isometrically embedded into a common metric space. This theorem is applied to define the notion of a converging sequence of points:

Definition 3.1. If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then one says $x_i \in X_i$ are a converging sequence that converge to $x_\infty \in \bar{X}_\infty$ if there exists a complete metric space Z and isometric embeddings $\varphi_i : X_i \rightarrow Z$ such that $\varphi_{i\#}T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#}T_\infty$ and $\varphi_i(x_i) \rightarrow \varphi_\infty(x_\infty)$. One says a collection of points, $\{p_{1,i}, p_{2,i}, \dots, p_{k,i}\}$, converges to a corresponding collection of points, $\{p_{1,\infty}, p_{2,\infty}, \dots, p_{k,\infty}\}$, if $\varphi_i(p_{j,i}) \rightarrow \varphi_\infty(p_{j,\infty})$ for $j = 1..k$.

Unlike in Gromov-Hausdorff convergence, there is a possibility of disappearing sequences of points:

Definition 3.2. If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then one says $x_i \in X_i$ are Cauchy if there exists a complete metric space Z and isometric embeddings $\varphi_i : M_i \rightarrow Z$ such that $\varphi_{i\#}T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#}T_\infty$ and $\varphi_i(x_i) \rightarrow z_\infty \in Z$. One says the sequence is disappearing if $z_\infty \notin \varphi_\infty(X_\infty)$. One says the sequence has no limit in \bar{X}_∞ if $z_\infty \notin \varphi_\infty(\bar{X}_\infty)$.

Remark 3.3. Examples with disappearing splines from [SW11] demonstrate that there exist Cauchy sequences of points which disappear. In fact z_∞ may not even lie in the metric completion of the limit space, $\varphi_\infty(\bar{X}_\infty)$.

Lemma 3.4. If a sequence of integral current spaces, $M_i = (X_i, d_i, T_i) \in \mathcal{M}_0^m$, converges to an integral current space, $M = (X, d, T) \in \mathcal{M}_0^m$, in the intrinsic flat sense, then every point x in the limit space X is the limit of points $x_i \in M_i$. In fact there exists a sequence of maps $F_i : X \rightarrow X_i$ such that $x_i = F_i(x)$ converges to x and

$$(109) \quad \lim_{i \rightarrow \infty} d_i(F_i(x), F_i(y)) = d(x, y) \quad \forall x, y \in X.$$

This sequence of maps F_i are not uniquely defined and are not even unique up to isometry.

¹Some of these notions were original defined in [Sor13] but they are now moved here.

Proof. By Theorem 2.31 there exists a common metric space Z and isometric embeddings $\varphi_i : X_i \rightarrow Z$ and $\varphi : X \rightarrow Z$ such that

$$(110) \quad \varphi_{\#}T - \varphi_{\#}T_i = U_i + \partial V_i$$

where $m_i = \mathbf{M}(U_i) + \mathbf{M}(V_i) \rightarrow 0$. So $\varphi_{\#}T_i$ converges in the flat and the weak sense to $\varphi_{\#}T$.

Let ρ_x be the distance function from $\varphi(x)$. Since $x \in \text{spt}(T)$, for any $\varepsilon > 0$,

$$(111) \quad \|\varphi_{\#}T\|(\rho_x^{-1}[0, \varepsilon]) > 0.$$

By the lower semicontinuity of mass,

$$(112) \quad \liminf_{i \rightarrow \infty} \|\varphi_{\#}T_i\|(\rho_x^{-1}[0, \varepsilon]) \geq \|\varphi_{\#}T\|(\rho_x^{-1}[0, \varepsilon]) > 0.$$

In particular,

$$(113) \quad \exists N_{\varepsilon, x} \in \mathbb{N} \text{ s.t. } \varphi_{\#}T \llcorner (\rho_x^{-1}[0, \varepsilon]) \neq 0 \quad \forall i \geq N_{\varepsilon, x}.$$

So for all $x \in X$ and any $j \in \mathbb{N}$

$$(114) \quad \exists N_{j, x} \text{ s.t. } \exists s_{i, j, x} \in \text{set}(\varphi_{\#}T) \cap B(x, 1/j) \quad \forall i \geq N_{j, x}.$$

Without loss of generality, assume $N_{j, x}$ is increasing in j . For $i \in \{1, \dots, N_{1, x}\}$ take $j_i = 1$. Then for $i \in \{N_{j-1, x} + 1, \dots, N_{j, x}\}$ let $j_i = j$. Thus $i \geq N_{j_i, x}$. Let

$$(115) \quad x_i = \varphi_i^{-1}(s_{i, j_i}).$$

Then $\varphi_i(x_i) \in B(x, 1/j_i)$ and $\varphi_i(x_i) \rightarrow \varphi(x)$.

Since this process can be completed for any $x \in X$, one has defined maps $F_i : X \rightarrow X_i$ such that

$$(116) \quad \varphi_i(F_i(x)) \rightarrow \varphi(x).$$

Finally, for all $x, y \in X$,

$$(117) \quad d_i(F_i(x), F_i(y)) = d_Z(\varphi_i(F_i(x)), \varphi_i(F_i(y))) \rightarrow d_Z(\varphi(x), \varphi(y)) = d(x, y).$$

□

Definition 3.5. Like any metric space, one can define the diameter of an integral current space, $M = (X, d, T)$, to be

$$(118) \quad \text{Diam}(M) = \sup \{d_X(x, y) : x, y \in X\} \in [0, \infty].$$

In addition, explicitly define the diameter of the 0 integral current space to be 0. A space is bounded if the diameter is finite.

Theorem 3.6. Suppose $M_i \xrightarrow{\mathcal{F}} M$ are integral current spaces then

$$(119) \quad \text{Diam}(M) \leq \liminf_{i \rightarrow \infty} \text{Diam}(M_i) \in [0, \infty]$$

Proof. Note that by the definition, $\text{Diam}(M_i) \geq 0$, so the liminf is always ≥ 0 . Thus the inequality is trivial when M is the 0 space. Assuming M is not the 0 space, for any $\varepsilon > 0$, there exists $x, y \in X$ such that

$$(120) \quad \text{Diam}(M) \leq d(x, y) + \varepsilon.$$

By Lemma 3.4, there exists $x_i, y_i \in X_i$ converging to $x, y \in X$ so that

$$(121) \quad \text{Diam}(M) \leq \lim_{i \rightarrow \infty} d_i(x_i, y_i) + \varepsilon \leq \liminf_{i \rightarrow \infty} \text{Diam}(X_i) + \varepsilon.$$

□

4. CONVERGENCE OF BALLS AND SPHERES

In this section the following key lemma concerning the convergence of balls and spheres is proven. It is an essential ingredient when trying to prove intrinsic flat limits are not the zero space or that points do not disappear. See Remark 4.2. It will be applied to prove Theorem 7.1, Theorem 8.1, and Example 9.1.

Lemma 4.1. *If $M_j = (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$ and $p_j \rightarrow p_\infty \in \bar{X}_\infty$, then there exists a subsequence of M_j also denoted M_j such that for almost every $r > 0$,*

$$(122) \quad S(p_j, r) = (\bar{B}(p_j, r), d_j, T_j \llcorner B(p_j, r))$$

are integral current spaces for $j \in \{1, 2, \dots, \infty\}$ and

$$(123) \quad S(p_j, r) \xrightarrow{\mathcal{F}} S(p_\infty, r).$$

If p_j are Cauchy with no limit in \bar{X}_∞ then there exists $\delta > 0$ such that for almost every $r \in (0, \delta)$ such that $S(p_j, r)$ are integral current spaces for $j \in \{1, 2, \dots\}$ and

$$(124) \quad S(p_j, r) \xrightarrow{\mathcal{F}} 0.$$

If $M_j \xrightarrow{\mathcal{F}} 0$ then for almost every r and for all sequences p_j one has (124).

In Example 4.3 demonstrates why it is necessary to choose a subsequence. Observe that this lemma does not require a uniform upper bound on volume and boundary volume.

Remark 4.2. *The first part of this lemma was stated as a lemma and applied by the author and Stefan Wenger to prove the intrinsic flat and Gromov-Hausdorff limits of noncollapsing sequences of Riemannian manifolds with nonnegative Ricci curvature agree in [SW11]. A reference to a proof of a related lemma by Ambrosio-Kirchheim [AK00] was provided there. As the lemma in [AK00] did allow for changing basepoints $p_j \neq p_\infty$, it was not completely clear to everyone how one should prove our lemma. So it is essential to provide full details here.*

Lemma 4.1 is now proven:

Proof. By Theorem 2.31 and 2.32 there exists a common complete metric space, Z , and isometric embeddings, $\varphi_j : X_j \rightarrow Z$ and $\varphi_\infty : X_\infty \rightarrow Z$, such that

$$(125) \quad \varphi_{j\#} T_j - T = \partial B_j + A_j$$

where $A_j \in \mathbf{I}_m(Z)$ and $B_j \in \mathbf{I}_{m+1}(Z)$ with

$$(126) \quad \mathbf{M}(A_j) + \mathbf{M}(B_j) \rightarrow 0$$

and where

$$(127) \quad T = \varphi_{\infty\#} T_\infty \in \mathbf{I}_m(Z) \text{ when } M_\infty \neq 0 \text{ and } T = 0 \text{ when } M_\infty = 0.$$

Since p_j are Cauchy,

$$(128) \quad z_j = \varphi_j(p_j) \rightarrow z_\infty \in Z.$$

When $p_j \rightarrow p_\infty$ then $z_\infty = \varphi_\infty(p_\infty)$. Then for almost every r

$$(129) \quad (\varphi_{j\#} T_j) \llcorner B(z_j, r) = \varphi_{j\#} S(p_j, r).$$

and

$$(130) \quad T \llcorner B(z_\infty, r) = \varphi_{\infty\#} S(p_\infty, r).$$

If p_j has no limit in \bar{X}_∞ , then $z_\infty \notin \varphi_\infty(\bar{X}_\infty)$ and so there exists $\delta > 0$ such that for all $r < \delta$,

$$(131) \quad B(z_\infty, r) \cap \varphi_\infty(\bar{X}_\infty) = \emptyset.$$

So

$$(132) \quad T \llcorner B(z_\infty, r) = \emptyset.$$

If $M_j \xrightarrow{\mathcal{F}} 0$, then one has this as well without requiring $r < \delta$.

To prove the theorem in all cases one need only show that for almost every r one can find a subsequence of the M_j also denoted M_j such that $S(p_j, r)$ are integral current spaces and

$$(133) \quad d_F^Z((\varphi_{j\#}T_j) \llcorner \rho_j^{-1}(-\infty, r), T \llcorner \rho_\infty^{-1}(-\infty, r)) \rightarrow 0$$

where $\rho_j(z) = d_Z(z_j, z)$.

By Lemma 2.34 for almost every r these are integral current spaces.

Observe that by (125), for almost every r :

$$(134) \quad (\varphi_{j\#}T_j) \llcorner \rho_j^{-1}(-\infty, r) - T \llcorner \rho_j^{-1}(-\infty, r) =$$

$$(135) \quad = (\partial B_j) \llcorner \rho_j^{-1}(-\infty, r) + A_j \llcorner \rho_j^{-1}(-\infty, r)$$

$$(136) \quad = \langle B_j, \rho_j, r \rangle + \partial(B_j \llcorner \rho_j^{-1}(-\infty, r)) + A_j \llcorner \rho_j^{-1}(-\infty, r).$$

Thus $d_F^Z((\varphi_{j\#}T_j) \llcorner \rho_j^{-1}(-\infty, r), T \llcorner \rho_j^{-1}(-\infty, r)) \leq$

$$(137) \quad \leq f_j(r) + \mathbf{M}(B_j \llcorner \rho_j^{-1}(-\infty, r)) + \mathbf{M}(A_j \llcorner \rho_j^{-1}(-\infty, r))$$

$$(138) \quad \leq f_j(r) + \mathbf{M}(B_j) + \mathbf{M}(A_j)$$

where

$$(139) \quad f_j(r) = \mathbf{M}(\langle B_j, \rho_j, r \rangle).$$

By the Ambrosio-Kirchheim Slicing Theorem

$$(140) \quad \int_{-\infty}^{\infty} f_j(r) dr = \int_{-\infty}^{\infty} \mathbf{M}(\langle B_j, \rho_j, r \rangle) dr$$

$$(141) \quad = \mathbf{M}(B_j \llcorner d\rho_j) \leq \text{Lip}(\rho_j) \mathbf{M}(B_j) \leq \mathbf{M}(B_j) \rightarrow 0.$$

Since f_j converge in L^1 to 0, there exists a subsequence, also denoted f_j , such that for almost every $r > 0$, $f_j(r)$ converge to 0 pointwise (c.f. [Rud87] Theorem 3.12).

Thus there is a subsequence such that for almost every $r > 0$

$$(142) \quad \lim_{j \rightarrow \infty} d_F^Z((\varphi_{j\#}T_j) \llcorner \rho_j^{-1}(-\infty, r), T \llcorner \rho_j^{-1}(-\infty, r)) = 0.$$

Next observe that the set

$$(143) \quad K = (\rho_j^{-1}(-\infty, r) \setminus \rho_\infty^{-1}(-\infty, r)) \cup (\rho_j^{-1}(-\infty, r) \setminus \rho_\infty^{-1}(-\infty, r))$$

satisfies

$$(144) \quad K \subset \rho_\infty^{-1}(r - \delta_j, r + \delta_j)$$

where

$$(145) \quad \delta_j = d_Z(z_j, z_\infty).$$

Then

$$\begin{aligned} d_F^Z(T \llcorner \rho_j^{-1}(-\infty, r), T \llcorner \rho_\infty^{-1}(-\infty, r)) &\leq \mathbf{M}(T \llcorner \rho_j^{-1}(-\infty, r) - T \llcorner \rho_\infty^{-1}(-\infty, r)) \\ &\leq \mathbf{M}(T \llcorner K) \\ &\leq \|T\|(\rho_\infty^{-1}(r - \delta_j, r + \delta_j)) \end{aligned}$$

Since $\lim_{j \rightarrow \infty} \delta_j = 0$, one has

$$(146) \quad \lim_{j \rightarrow \infty} \|T\|(\rho_\infty^{-1}(r - \delta_j, r + \delta_j)) = \lim_{j \rightarrow \infty} \|f_\# T\|(\rho_\infty^{-1}(r - \delta_j, r + \delta_j))$$

$$(147) \quad = \|f_\# T\|(\{r\})$$

Since $\|f_\# T\|$ is a finite measure on \mathbb{R} , $\|f_\# T\|(\{r\}) = 0$ except on a countable set of values of r . Thus, for almost every r ,

$$(148) \quad \lim_{j \rightarrow \infty} d_F^Z(T \llcorner \rho_j^{-1}(-\infty, r), T \llcorner \rho_\infty^{-1}(-\infty, r)) = 0.$$

Combining this with (142) one has (133) and the proof is complete. \square

Example 4.3. *There exists a sequence of Riemannian manifolds M_j diffeomorphic to a torus with $\text{vol}(M_j) \leq V_0$ such that $M_j \xrightarrow{\mathcal{F}} 0$ but there exists a Cauchy sequence $p_j \in M_j$ such that $S(p_j, r)$ does not have an intrinsic flat limit for any $r \in (0, \pi)$.*

Proof. Take the metric

$$(149) \quad g_j = dr^2 + f_j^2(r) d\theta^2 \quad r \in [0, \pi]$$

with $f_j(0) = 0$, $f_j(\pi) = 0$, $f_j'(0) = 1$, $f_j'(\pi) = -1$ so that M_j is a smooth Riemannian manifold. Choose $f_j > 0$ smooth on $(0, \pi)$ such that

$$(150) \quad \int_0^\pi f_j^2(r) dr \rightarrow 0$$

and such that

$$(151) \quad f_j(r) > 1 \text{ for } r \in [j \bmod \pi, j + 1/j \bmod \pi] \cap (1/j^2, \pi - 1/j^2)$$

and

$$(152) \quad f_j(r) < 1/j \text{ for } r \in [j + 2/j \bmod \pi, j + 3/j \bmod \pi] \cap (1/j^2, \pi - 1/j^2)$$

and f_j smoothly decreasing in between. Since

$$(153) \quad \text{Vol}(M_j) = 4\pi \int_0^{2\pi} f_j^2(r) dr \rightarrow 0$$

one has $M_j \xrightarrow{\mathcal{F}} \mathbf{0}$. Take p_j to be the point where $r = 0$. Suppose one has r' such that the balls converge to the zero integral current space, $S(p_j, r') \xrightarrow{\mathcal{F}} \mathbf{0}$, then the spheres also converge to the zero space, $\partial S(p_j, r') \xrightarrow{\mathcal{F}} \mathbf{0}$.

However there exists a subsequence $j' \rightarrow \infty$ such that $r \in [j' \bmod \pi, j' + 1/j' \bmod \pi]$. On this set $S(p_{j'}, r)$ is bi-Lipschitz close to a circle S^1 endowed with the restricted metric from the disk. So

$$(154) \quad \partial S(p_{j'}, r) \xrightarrow{\mathcal{F}} (S^1, d_{D^2}, \int_{S^1}).$$

\square

Also useful for some applications is the following lemma:

Lemma 4.4. *Let $M_j = (X_j, d_j, T_j)$ and let $R > 0$. Then we have rescaled integral current spaces, $M'_j = (X_j, d_j/R, T_j)$, one of which may possibly be $\mathbf{0}$, and*

$$(155) \quad d_{\mathcal{F}}(M_1, M_2) \leq d_{\mathcal{F}}(M'_1, M'_2)R^m(1+R).$$

In particular taking almost any $r = R \in (0, \delta)$ and $p_j \in X_j$ one can rescale

$$(156) \quad S(p_j, r) = (\text{set}(T_j \llcorner B(p_j, r)), d_j, T_j \llcorner B(p_j, r))$$

by r to obtain

$$(157) \quad S'(p_j, 1) = (\text{set}(T_j \llcorner B(p_j, 1)), d_j/R, T_j \llcorner B(p_j, 1))$$

and

$$(158) \quad d_{\mathcal{F}}(S(p_1, r), S(p_2, r)) \leq d_{\mathcal{F}}(S'(p_1, 1), S'(p_2, 1))r^m(1+\delta).$$

Proof. By the Theorem 2.28, there exists isometric embeddings $\varphi_j : X_j \rightarrow Z$

$$(159) \quad d_Z(\varphi_j(x), \varphi_j(y))/R = d_j(x, y)/R \quad \forall x, y \in X_j$$

and $A \in \mathbf{I}_m(Z)$, $B \in \mathbf{I}_{m+1}(Z)$ such that

$$(160) \quad \varphi_{1\#}T_1 - \varphi_{2\#}T_2 = A + \partial B$$

and

$$(161) \quad d_{\mathcal{F}}(M'_1, M'_2) = \mathbf{M}(A) + \mathbf{M}(B)$$

where these masses are defined using d_Z/R . Then $\varphi_j : X_j \rightarrow Z$

$$(162) \quad d_Z(\varphi_j(x), \varphi_j(y)) = d_j(x, y) \quad \forall x, y \in X_j$$

and so by definition of intrinsic flat distance

$$(163) \quad d_{\mathcal{F}}(M_1, M_2) \leq \mathbf{M}'(A) + \mathbf{M}'(B)$$

where these masses are defined using d_Z . Thus

$$(164) \quad d_{\mathcal{F}}(M_1, M_2) \leq \mathbf{M}(A)R^m + \mathbf{M}(B)R^{m+1}$$

$$(165) \quad \leq (\mathbf{M}(A) + \mathbf{M}(B))R^m(1+R)$$

$$(166) \quad \leq d_{\mathcal{F}}(M'_1, M'_2)R^m(1+R).$$

It is easy to see this argument also works when $M_2 = \mathbf{0}$ taking $\varphi_{2\#}T_2 = 0$. \square

5. FLAT CONVERGENCE TO GROMOV-HAUSDORFF CONVERGENCE

In this subsection, Theorem 5.1 is proven:

Theorem 5.1. ² *If a sequence of precompact integral current spaces, $M_i = (X_i, d_i, T_i) \in \mathcal{M}_0^m$, converges to a nonzero precompact integral current space, $M = (X, d, T) \in \mathcal{M}_0^m$, in the intrinsic flat sense, then there exists $S_i \in \mathbf{I}_m(\bar{X}_i)$ such that $N_i = (\text{set}(S_i), d_i)$ converges to (\bar{X}, d) in the Gromov-Hausdorff sense*

$$(167) \quad d_{GH}(N_i, M) \rightarrow 0$$

and

$$(168) \quad \liminf_{i \rightarrow \infty} \mathbf{M}(S_i) \geq \mathbf{M}(M).$$

²This theorem and its proof originally appeared an early preprint version of [Sor13] but has now been moved to this paper with minor corrections. It will not appear in any publication of [Sor13].

When the M_i are Riemannian manifolds, the N_i can be taken to be settled completions of open submanifolds of M_i .

Remark 5.2. If in addition it is assumed that $\lim_{i \rightarrow \infty} \mathbf{M}(M_i) = \mathbf{M}(M)$, then by (168),

$$(169) \quad \lim_{i \rightarrow \infty} \mathbf{M}(\text{set}(T_i - S_i), d_i, T_i - S_i) = 0.$$

In the Riemannian setting,

$$(170) \quad \lim_{i \rightarrow \infty} \text{Vol}(M_i \setminus N_i) = 0.$$

Remark 5.3. In Ilmanen's example [SW11] of a sphere with increasingly many splines, the S_i may be chosen to be integration over the spherical part of M_i with balls around the tips removed. Then $\text{set}(S_i)$ are manifolds with boundary converging to the sphere in the Gromov-Hausdorff and intrinsic flat sense.

Remark 5.4. The precompactness of the limit integral current spaces is necessary in this theorem because a noncompact limit space can never be the Gromov-Hausdorff limit of precompact spaces. In fact there are sequences of compact Riemannian manifolds, M_j , whose intrinsic flat limit is an unbounded complete Riemannian manifold of finite volume [SW11][Ex A.10] and another example of such spaces whose Intrinsic Flat limit is a bounded noncompact integral current space [SW11][Ex A.11].

Remark 5.5. Gromov's Compactness Theorem combined with Theorem 5.1 implies that that any sequence of $x_i \in N_i \subset M_i$ has a subsequence converging to a point x in the metric completion of M . Other points need not have limit points, as can be seen when the tips of thin splines disappear in the examples from [SW11]. A more general Bolzano-Weierstrass Theorem precisely identifying those points which do not disappear is proven later in this section.

Theorem 5.1 is now proven:

Proof. By Theorem 2.31 there exists a common metric space Z and isometric embeddings $\varphi_i : X_i \rightarrow Z$ and $\varphi : X \rightarrow Z$ such that

$$(171) \quad \varphi_{\#} T - \varphi_{i\#} T_i = U_i + \partial V_i$$

where $m_i = \mathbf{M}(U_i) + \mathbf{M}(V_i) \rightarrow 0$. So $\varphi_{i\#} T_i$ converges in the flat and thus the weak sense to $\varphi_{\#} T$.

Since $M \in \mathcal{M}_0^m$, $\varphi(X)$ is precompact. Let $\rho : Z \rightarrow \mathbb{R}$ be the distance function from $\varphi(X)$.

By the Ambrosio-Kirchheim Slicing Theorem [Theorem 2.18],

$$(172) \quad S_{i,\epsilon} := \varphi_{i\#} T_i \llcorner \rho^{-1}([0, \epsilon]) \in \mathbf{I}_m(Z)$$

for almost every $\epsilon > 0$. Fix any such ϵ .

Before choosing the S_i mentioned in the statement of the theorem, one may examine the mass of $S_{i,\epsilon}$ and the Hausdorff distance between $\text{set}(S_{i,\epsilon})$ and $\varphi(X)$. Note that $\varphi_{\#} T = \varphi_{\#} T \llcorner \rho^{-1}([0, \epsilon])$. So

$$(173) \quad \|T\|(\rho^{-1}[0, \delta]) = \mathbf{M}(T).$$

By lower semicontinuity of mass we have

$$(174) \quad \liminf_{i \rightarrow \infty} \|\varphi_{i\#} T_i\|(\rho^{-1}[0, \epsilon]) \geq \|\varphi_{\#} T\|(\rho^{-1}[0, \epsilon]).$$

Combining this with (172) and (173) and the definition of \liminf one has:

$$(175) \quad \text{for almost every } \epsilon > 0 \exists N'_\epsilon \in \mathbb{N} \text{ such that } \mathbf{M}(S_{i,\epsilon}) \geq \mathbf{M}(T) - \epsilon \quad \forall i \geq N'_\epsilon.$$

To see that the Hausdorff distance between $S_{i,\epsilon}$ and $\varphi(X)$ is small, $d_H^Z(S_{i,\epsilon}, \varphi(X)) < 2\epsilon$, first immediately observe that

$$(176) \quad \text{set}(S_{i,\epsilon}) \subset \bar{T}_\epsilon(\varphi(X)) \subset T_{2\epsilon}(\varphi(X)).$$

One needs only show

$$(177) \quad \varphi(X) \subset T_{2\epsilon}(\text{set}(S_{i,\epsilon})) \quad \forall i \geq N_\epsilon.$$

To prove (177), first note that for any $x \in X$, one can let ρ_x be the distance function from $\varphi(x)$. By the lower semicontinuity of mass

$$(178) \quad \liminf_{i \rightarrow \infty} \|\varphi_{\#} T_i\|(\rho_x^{-1}[0, \epsilon]) \geq \|\varphi_{\#} T\|(\rho_x^{-1}[0, \epsilon]) > 0 \quad \forall \epsilon > 0.$$

Thus one has

$$(179) \quad \text{for almost every } \epsilon > 0 \exists N_{\epsilon,x} \geq N'_\epsilon \text{ s.t. } \varphi_{\#} T_i \llcorner \rho_x^{-1}[0, \epsilon] \neq 0 \quad \forall i \geq N_{\epsilon,x}.$$

Recall N'_ϵ was defined in (175). Combining this with (172), and the fact that

$$(180) \quad \rho_x^{-1}[0, \epsilon] = B(x, \epsilon) \subset \rho^{-1}[0, \epsilon] = T_\epsilon(\varphi(X))$$

we have

$$(181) \quad \forall x \in X \text{ for almost every } \epsilon > 0 \exists N_{\epsilon,x} \geq N'_\epsilon \text{ and } s_{i,\epsilon,x} \in \text{set}(S_i) \cap B(\varphi(x), r).$$

By the precompactness of X , there is a finite ϵ net, $X_\epsilon = \{x_1, \dots, x_N\}$ on $\varphi(X)$ (i.e. the union of $B(x_i, \epsilon)$ contains X_ϵ). Define

$$(182) \quad N_\epsilon = \max \{N_{\epsilon,x_j} : x_j \in X_\epsilon\} \geq N'_\epsilon$$

then

$$(183) \quad \forall x \in X \exists x_j \in X_\epsilon \text{ s.t. } \forall i \geq N_\epsilon \exists s_x := s_{i,\epsilon,x_j} \in \text{set}(S_{i,\epsilon}) \text{ s.t. } d_Z(s_x, \varphi(x)) < 2\epsilon.$$

So (177) has been proven.

Combining (177) with (176), the Hausdorff distance satisfies

$$(184) \quad d_H^Z(\text{set}(S_{i,\epsilon}), \varphi(X)) \leq 2\epsilon \quad \forall i \geq N_\epsilon.$$

We now define S_i in the statement of the theorem and prove (167) and (168).

Let $\epsilon_k \rightarrow 0$ be a decreasing sequence of ϵ for which all these currents are defined. Let $N_k := N_{\epsilon_k}$. Let

$$(185) \quad S_i = T_i \in \mathbf{I}_m(X_i) \text{ for } i = 1 \text{ to } N_1$$

$$(186) \quad S_i = \varphi_{i\#}^{-1} S_{i,\epsilon_1} \in \mathbf{I}_m(X_i) \text{ for } i = N_1 + 1 \text{ to } N_2$$

and so on:

$$(187) \quad S_i = \varphi_{i\#}^{-1} S_{i,\epsilon_j} \in \mathbf{I}_m(X_i) \text{ for } i = N_j + 1 \text{ to } N_{j+1}$$

Then by (184),

$$(188) \quad d_H^Z(\text{set}(S_i), \varphi(X)) \leq 2\epsilon_i.$$

This implies (167).

By (175) and $N_k = N_{\epsilon_k} \geq N'_{\epsilon_k}$ one has, we have

$$(189) \quad \mathbf{M}(S_i) \geq \mathbf{M}(T) - \epsilon_i$$

which gives us (168) and completes the proof of the theorem. \square

Remark 5.6. *One could construct a common metric space Z for Examples A.10 and A.11 of [SW11] and find $S_{i,\epsilon}$ as in the above proof satisfying (176). However, in that example, (177) will fail to hold. This is where the precompactness of the limit space is essential in the proof.*

Remark 5.7. *Examples in [SW11] demonstrate that the metric space of a current space need not be a length space. In general, when a sequence of Riemannian manifolds converges in the intrinsic flat sense to an integral current space it need not be a geodesic length space. If the set (S_i) are length spaces or approximately length spaces, then the limit current space is in fact a length space. This occurs for example in Ilmanen's example of [SW11]. It also occurs whenever the Gromov-Hausdorff limits and flat limits of length spaces agree. It might be interesting to develop a notion of an approximate length space that suffices to give a geodesic limit space. What properties must hold on M_i to guarantee that their limit is a geodesic length space?*

Remark 5.8. *It is not immediately clear whether the integral current spaces, N_i , constructed in the proof of Theorem 5.1 actually converge in the intrinsic flat sense to M . One expects an extra assumption on total mass would be needed to interchange between flat and weak convergence, but even so it is not completely clear. One would need to uniformly control the masses of ∂N_i using a common upper bound on $\mathbf{M}(N)$ which can be done using theorems in Section 5 of [AK00], but is highly technical. It is only worth investigating if one has an application in mind.*

6. ARZELA-ASCOLI THEOREM FOR LIPSCHITZ FUNCTIONS

In this section our first Arzela-Ascoli Theorem is proven. This basic theorem is proven using only Theorem 2.31 and Lemma 3.4.

Theorem 6.1. ³ Fix $K > 0$. Suppose $M_i = (X_i, d_i, T_i)$ are integral current spaces and $M_i \xrightarrow{\mathcal{F}} M_\infty$ and $F_i : X_i \rightarrow W$ are Lipschitz maps into a compact metric space W with

$$(190) \quad \text{Lip}(F_i) \leq K,$$

then a subsequence converges to a Lipschitz map $F_\infty : X_\infty \rightarrow W$ with

$$(191) \quad \text{Lip}(F_\infty) \leq K.$$

More specifically, there exists isometric embeddings of the subsequence, $\varphi_i : X_i \rightarrow Z$, such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \rightarrow 0$ and for any sequence $p_i \in X_i$ converging to $p \in X_\infty$,

$$(192) \quad d_Z(\varphi_i(p_i), \varphi_\infty(p)) \rightarrow 0,$$

one has converging images,

$$(193) \quad d_W(F_i(p_i), F_\infty(p)) \rightarrow 0.$$

Proof. By Theorem 2.31, $\varphi_i : M_i \rightarrow Z$ such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \rightarrow 0$.

Take any $p_\infty \in X_\infty$. By Lemma 3.4, there exists $p_i \in X_i$ such that $\lim_{i \rightarrow \infty} \varphi_i(p_i) = \varphi_\infty(p_\infty)$. Their images $F_i(p_i) \in W$ have a subsequence which converges to some $w \in W$. Set $F_\infty(p_\infty) = w$. Recall that integral current spaces are separable. So there is a countable dense subset $X_0 \subset X_\infty$. Thus one may repeat this process creating subsequences

³This theorem originally appeared with a fundamentally different more difficult proof involving Gromov filling volumes in an early preprint version of [Sor13]. It will not appear in any publication of [Sor13].

of subsequences for a countable dense collection of $p \in X_0 = X_\infty$. Diagonalizing, one obtains the subsequence mentioned in the theorem statement and a function,

$$(194) \quad F_\infty : X_0 \subset X_\infty \rightarrow W.$$

We need to extend F_∞ to define a limit function from X to W . Observe that for all $p, q \in X_0$ there exists p_i and q_i converging to them such that

$$(195) \quad d_W(F_\infty(p), F_\infty(q)) = \lim_{i \rightarrow \infty} d_W(F_i(p_i), F_i(q_i))$$

$$(196) \quad \leq \lim_{i \rightarrow \infty} K d_{X_i}(p_i, q_i)$$

$$(197) \quad \leq \lim_{i \rightarrow \infty} K d_Z(\varphi_i(p_i), \varphi_i(q_i))$$

$$(198) \quad \leq K d_Z(\varphi_\infty(p), \varphi_\infty(q))$$

$$(199) \quad \leq K d_{X_\infty}(p, q).$$

Thus one may extend F_∞ continuously to

$$(200) \quad F_\infty : X_\infty \rightarrow W \text{ and } \text{Lip}(F_\infty) \leq K.$$

Now suppose we have $p_i \rightarrow p$ as in (192). We must prove (193). Assume on the contrary that there exists a subsequence of p_i also denoted p_i such that

$$(201) \quad \exists r_0 > 0 \text{ s.t. } d_W(F_i(p_i), F_\infty(p)) > r_0.$$

By (192), there exists $N_0 \in \mathbb{N}$ such that

$$(202) \quad d_Z(\varphi_i(p_i), \varphi_\infty(p)) < r_0/10 \quad \forall i \geq N_0.$$

By the definition of the continuous extension, there exists $q_j \in X_0$ and there exists $N_1 \in \mathbb{N}$ such that

$$(203) \quad d_Z(\varphi_\infty(q_j), \varphi_\infty(p)) = d_X(q_j, p) < r_0/(10K) \quad \forall j \geq N_1$$

and

$$(204) \quad d_W(F_\infty(q_j), F_\infty(p)) \leq K d_Z(\varphi_\infty(q_j), \varphi_\infty(p)) = K d_X(q_j, p) < r_0/10 \quad \forall j \geq N_1.$$

By the definition of $F_\infty : X_0 \rightarrow W$, for each fixed j , there exists $q_{j,i} \in X_i$ and $N_j, N'_j \in \mathbb{N}$ with

$$(205) \quad d_Z(\varphi_i(q_{j,i}), \varphi_\infty(q_j)) < r_0/(10K) \quad \forall i \geq N_j$$

and

$$(206) \quad d_W(F_i(q_{j,i}), F_\infty(q_j)) \leq r_0/5 \quad \forall i \geq N'_j.$$

Also $\text{Lip}(F_i) \leq K$ implies:

$$(207) \quad d_W(F_i(p_i), F_i(q_{j,i})) \leq K d_{X_i}(p_i, q_{j,i}).$$

Take any $j \geq N_1$ and any $i \geq \max\{N'_j, N_j, N_0\}$. By (192), (203) and (205) we have

$$\begin{aligned} d_{X_i}(p_i, q_{j,i}) &= d_Z(\varphi_i(p_i), \varphi_i(q_{j,i})) \\ &\leq d_Z(\varphi_i(p_i), \varphi_\infty(p)) + d_Z(\varphi_\infty(p), \varphi_\infty(q_j)) + d_Z(\varphi_i(q_{j,i}), \varphi_\infty(q_j)) \\ &\leq 3r_0/(10K) \end{aligned}$$

Combining this with (204), (206) and (207), we have

$$\begin{aligned} d_W(F_i(p_i), F_\infty(p)) &\leq d_W(F_i(p_i), F_i(q_{j,i})) + d_W(F_i(q_{j,i}), F_\infty(q_j)) + d_W(F_\infty(q_j), F_\infty(p)) \\ &\leq K d_{X_i}(p_i, q_{j,i}) + d_W(F_i(q_{j,i}), F_\infty(q_j)) + K d_X(q_j, p) \\ &\leq K(3r_0/(10K)) + (r_0/5) + r_0/10 = 6r_0/10 < r_0, \end{aligned}$$

which is a contradiction. \square

Remark 6.2. Recall that the corresponding Gromov-Hausdorff Arzela-Ascoli Theorem allows the target spaces to vary as well: $F_i : X_i \rightarrow W_i$ of Lipschitz maps into compact metric spaces W_i with $\text{Lip}(F_i) \leq K$ where $W_i \xrightarrow{GH} W$ and $X_i \xrightarrow{GH} X$. See for example Grove-Petersen [GP91]. The corresponding statement allowing both $X_i \xrightarrow{\mathcal{F}} X$ and $W_i \xrightarrow{\mathcal{F}} W$ is false. For example, one may have a sequence of compact connected manifolds, W_i , which converge in the intrinsic flat sense to a compact metric space, W , that is not connected [SW11]. In that setting one has a sequence of Lipschitz maps which are unit speed geodesics, $F_i : [0, 1] \rightarrow W_i$ where $W_i \xrightarrow{\mathcal{F}} W$ with no limiting function $F : [0, 1] \rightarrow \bar{W}_\infty$.

Remark 6.3. It should be possible to extend Theorem 6.1 to sequences $F_i : X_i \rightarrow W_i$ of Lipschitz maps into compact metric spaces W_i with $\text{Lip}(F_i) \leq K$ where $W_i \xrightarrow{GH} W$ and $X_i \xrightarrow{\mathcal{F}} X$ using Gromov's Embedding Theorem or the work of Grove-Petersen [GP91]. No applications are known for such a theorem at this time so there is no need to prove this here.

7. BASIC BOLZANO-WEIERSTRASS THEOREM

In this section, Theorem 7.1 is proven. Recall Lemma 2.34 states that for almost every radius $S(p, r)$ of (73) is an integral current space. Recall also that, like any integral current space, $d_{\mathcal{F}}(S(p, r), \mathbf{0}) = 0$ iff $S(p, r) = \mathbf{0}$ [SW10]. If one considers a sequence of integral current spaces, M_i with points p_i , then for almost every r , $S(p_i, r)$ is an integral current space for all i in the sequence. In our basic Bolzano-Weierstrass Theorem we assume these $S(p_i, r)$ are kept a definite distance away from $\mathbf{0}$ where this distance depends upon on the radius. A different Bolzano-Weierstrass Theorem which involves the Gromov Filling Volume appears in [Sor13].

Theorem 7.1. Suppose $M_i^m = (X_i, d_i, T_i)$ are integral current spaces which converge in the intrinsic flat sense to a nonzero integral current space $M_\infty^m = (X_\infty, d_\infty, T_\infty)$. Suppose there exists $r_0 > 0$, a positive function $h : (0, r_0) \rightarrow (0, r_0)$, and a sequence $p_i \in M_i$ such that for almost every $r \in (0, r_0)$

$$(208) \quad \liminf_{i \rightarrow \infty} d_{\mathcal{F}}(S(p_i, r), \mathbf{0}) \geq h(r) > 0.$$

Then there exists a subsequence, also denoted M_i , such that p_i converges to $p_\infty \in \bar{X}_\infty$.

Remark 7.2. Note that M_i and M_∞ are not required to be precompact. The M_i are not required to have uniformly bounded mass or volume. The key hypothesis is that the $M_i \xrightarrow{\mathcal{F}} M_\infty$ and that M_∞ has finite mass. For this reason there is not enough room to fit too many balls of mass $h(r)$ in M_∞ . This allows us to produce a converging subsequence in the style of a classical Bolzano-Weierstrass Theorem.

Remark 7.3. It is possible that $p_\infty \notin X_\infty$ as can be seen by taking all the $M_i = M_\infty$ a manifold M with a cusp singularity at p_∞ so that $M_\infty = M \setminus p_\infty$ and p_i a sequence of points approaching p_∞ .

Proof. By Theorem 2.31 there exists a common metric space Z and isometric embeddings $\varphi_j : X_j \rightarrow Z$ and $\varphi_\infty : X_\infty \rightarrow Z$ such that

$$(209) \quad \varphi_{j\#} T_j - T = \partial B_j + A_j$$

where $A_j \in \mathbf{I}_m(Z)$ and $B_j \in \mathbf{I}_{m+1}(Z)$ with

$$(210) \quad \mathbf{M}(A_j) + \mathbf{M}(B_j) \rightarrow 0$$

and where

$$(211) \quad T = \varphi_{\infty\#} T_\infty \in \mathbf{I}_m(Z).$$

One needs only show that a subsequence of $\varphi_i(p_i)$ is a Cauchy sequence. Once this is done, one can apply Lemma 4.1 to the subsequence. In that lemma, it is shown that a Cauchy sequence, p_i , converges to $p_\infty \in \bar{X}_\infty$ unless there is a radius r sufficiently small that $S(p_i, r) \xrightarrow{\mathcal{F}} \mathbf{0}$. Since this is not allowed by the hypothesis of the theorem being proven, one sees that the subsequence converges to $p_\infty \in \bar{X}_\infty$ as desired.

So one needs only prove that a subsequence $\varphi_i(p_i)$ converges in Z . This is not immediate because Z is only complete and need not be compact.

Assume on the contrary that

$$(212) \quad \exists \delta > 0 \text{ s.t. } d_Z(\varphi_i(p_i), \varphi_j(p_j)) \geq \delta \quad \forall i, j \in \mathbb{N}.$$

Let $\rho_i(x) = d_Z(\varphi_i(p_i), x)$, then for almost every $r \in (0, r_0) \cap (0, \delta/2)$,

$$(213) \quad \rho_i^{-1}(-\infty, r) \cap \rho_j^{-1}(-\infty, r) = \emptyset \quad \forall i, j \in \mathbb{N}.$$

Now

$$(214) \quad (\varphi_{i\#} T_i) \llcorner \rho_j^{-1}(-\infty, r) - \varphi_{\infty\#} T_\infty \llcorner \rho_j^{-1}(-\infty, r) =$$

$$(215) \quad = (\partial B_i) \llcorner \rho_j^{-1}(-\infty, r) + A_i \llcorner \rho_j^{-1}(-\infty, r)$$

$$(216) \quad = \langle B_i, \rho_j, r \rangle + \partial(B_i \llcorner \rho_j^{-1}(-\infty, r)) + A_i \llcorner \rho_j^{-1}(-\infty, r).$$

Thus $d_F^Z(\varphi_{i\#} T_i \llcorner \rho_j^{-1}(-\infty, r), \varphi_{\infty\#} T_\infty \llcorner \rho_j^{-1}(-\infty, r)) \leq$

$$(217) \quad \leq f_{ij}(r) + \mathbf{M}(B_i \llcorner \rho_j^{-1}(-\infty, r)) + \mathbf{M}(A_i \llcorner \rho_j^{-1}(-\infty, r))$$

$$(218) \quad \leq f_{ij}(r) + \mathbf{M}(B_i) + \mathbf{M}(A_i)$$

where

$$(219) \quad f_{ij}(r) = \mathbf{M}(\langle B_i, \rho_j, r \rangle).$$

By the Ambrosio-Kirchheim Slicing Theorem, for fixed $j \in \mathbb{N}$,

$$(220) \quad \int_{-\infty}^{\infty} f_{ij}(r) dr = \int_{-\infty}^{\infty} \mathbf{M}(\langle B_i, \rho_j, r \rangle) dr$$

$$(221) \quad = \mathbf{M}(B_i \llcorner d\rho_j) \leq \text{Lip}(\rho_j) \mathbf{M}(B_i) \leq \mathbf{M}(B_i)$$

which converges to 0 as $i \rightarrow \infty$. Thus for fixed j and almost every r there is a subsequence $i' \rightarrow \infty$ such that $\lim_{i' \rightarrow \infty} f_{i'j}(r) = 0$ pointwise. Diagonalizing, there is a subsequence i'' such that for all j , $\lim_{i'' \rightarrow \infty} f_{i''j}(r) = 0$ pointwise.

Thus for almost every $r \in (0, r_0) \cap (0, \delta/2)$, there is a subsequence i'' such that for all $j \in \mathbb{N}$,

$$(222) \quad d_F^Z(\varphi_{i''\#} T_{i''} \llcorner \rho_j^{-1}(-\infty, r), \varphi_{\infty\#} T_\infty \llcorner \rho_j^{-1}(-\infty, r)) \rightarrow 0$$

Since the balls are disjoint,

$$(223) \quad \mathbf{M}(T_\infty) \geq \sum_{j=1}^N \mathbf{M}(\varphi_{\infty\#} T_\infty \llcorner \rho_j^{-1}(-\infty, r)).$$

Thus

$$(224) \quad \limsup_{j \rightarrow \infty} \mathbf{M}(\varphi_{\infty\#} T_{\infty} \llcorner \rho_j^{-1}(-\infty, r)) = 0.$$

So

$$(225) \quad \limsup_{j \rightarrow \infty} d_F^Z(\varphi_{\infty\#} T_{\infty} \llcorner \rho_j^{-1}(-\infty, r), \mathbf{0}) = 0.$$

In particular, for j sufficiently large

$$(226) \quad d_F^Z(\varphi_{\infty\#} T_{\infty} \llcorner \rho_j^{-1}(-\infty, r), \mathbf{0}) < h(r)/2.$$

Combining this with (222), for i sufficiently large

$$(227) \quad d_{\mathcal{F}}(S(p_i, r), \mathbf{0}) \leq d_F^Z(\varphi_{i\#} T_i \llcorner \rho_i^{-1}(-\infty, r), \mathbf{0}) < h(r)/2$$

which contradicts the hypothesis. Thus there is a subsequence $\varphi_i(p_i)$ which converges to some point $z_{\infty} \in Z$ exactly as needed. \square

8. LIMITS OF UNIFORMLY LOCAL ISOMETRIES

In this section as Arzela-Ascoli Theorem which allows both the domain and the target spaces to converge in the intrinsic flat sense. This theorem applies to sequences of oriented Riemannian manifolds M_i with

$$(228) \quad \text{Vol}(M_i) \leq V_i \text{ and } \text{Vol}(\partial M_i) \leq A_i$$

and functions $F_i : M_i \rightarrow M'_i$ which are orientation preserving local isometries that are isometries on balls of a fixed radius, $\delta > 0$ which is uniform for the sequence.

Theorem 8.1. ⁴ *Let $M_i = (X_i, d_i, T_i)$ and $M'_i = (X'_i, d'_i, T'_i)$ be integral current spaces such that*

$$(229) \quad M_i \xrightarrow{\mathcal{F}} M_{\infty} \text{ and } M'_i \xrightarrow{\mathcal{F}} M'_{\infty}.$$

Fix $\delta > 0$. Let $F_i : M_i \rightarrow M'_i$ be continuous maps which are current preserving isometries on balls of radius δ in the sense that:

$$(230) \quad \forall x \in X_i, F_i : \bar{B}(x, \delta) \rightarrow \bar{B}(F_i(x), \delta) \text{ is an isometry}$$

and

$$(231) \quad F_{i\#}(T_i \llcorner B(x, r)) = T'_i \llcorner B(F_i(x), r) \text{ for almost every } r \in (0, \delta).$$

Then, when $M_{\infty} \neq \mathbf{0}$, one has $M'_{\infty} \neq \mathbf{0}$ and there is a subsequence, also denoted F_i , which converges to a (surjective) local isometry

$$(232) \quad F_{\infty} : \bar{X}_{\infty} \rightarrow \bar{X}'_{\infty}.$$

More specifically, there exists isometric embeddings of the subsequence $\varphi_i : X_i \rightarrow Z$, $\varphi'_i : X'_i \rightarrow Z'$, such that

$$(233) \quad d_F^Z(\varphi_{i\#} T_i, \varphi_{\infty\#} T_{\infty}) \rightarrow 0 \text{ and } d_F^{Z'}(\varphi'_{i\#} T'_i, \varphi'_{\infty\#} T'_{\infty}) \rightarrow 0$$

and for any sequence $p_i \in X_i$ converging to $p \in X_{\infty}$:

$$(234) \quad \lim_{i \rightarrow \infty} \varphi_i(p_i) = \varphi_{\infty}(p) \in Z$$

⁴A similar theorem with slightly different hypothesis originally appeared with a fundamentally different more difficult proof involving Gromov filling volumes in an early preprint version of [Sor13]. That will not appear in any publication of [Sor13] as this theorem is better.

one has

$$(235) \quad \lim_{i \rightarrow \infty} \varphi'_i(F_i(p_i)) = \varphi'_\infty(F_\infty(p_\infty)) \in Z'.$$

When $M_\infty = \mathbf{0}$ and F_i are surjective, one has $M'_\infty = \mathbf{0}$.

Remark 8.2. Example 8.5 describes the necessity of the uniformity condition (230) in Theorem 8.1.

Remark 8.3. It may be possible to prove that the limit map here is also current preserving on balls of radius less than δ . This is technical and not needed for our applications. So we do not prove this here.

Remark 8.4. It may be possible to prove a similar theorem replacing the surjective uniformly local isometries with surjective uniformly local uniformly bi-Lipschitz maps but the proof would be fairly technical and there is no immediate application for this at this time.

Theorem 8.1 is now proven:

Proof. By Theorem 2.31 there exists $\varphi_i : M_i \rightarrow Z$ such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \rightarrow 0$ and $\varphi'_i : M'_i \rightarrow Z'$ such that $d_F^{Z'}(\varphi'_{i\#}T'_i, \varphi'_\infty T'_\infty) \rightarrow 0$.

Assuming $M'_\infty \neq \mathbf{0}$, one must first find a subsequence and construct the limit function $F_\infty : P \rightarrow X'_\infty$ satisfying (235) for all $p \in P$ where P is a countably dense collection of points in X_∞ .

Take any $p \in P$. Recall $S(p, r) = (\text{set}(T_\infty \llcorner B(p, r)), d_\infty, T_\infty \llcorner B(p, r))$ is defined for almost every r . Since $p \in X_\infty$, and $X_\infty = \text{set}(T_\infty)$,

$$(236) \quad \liminf_{r \rightarrow 0} \mathbf{M}(S(p, r))/r^m = \liminf_{r \rightarrow 0} \|T_\infty\|(B(p, r))/r^m > 0.$$

In particular

$$(237) \quad S(p, r) \neq \mathbf{0}.$$

By Lemma 3.4 there exists $p_i \in X_i$ such that

$$(238) \quad \lim_{i \rightarrow \infty} \varphi_i(p_i) = \varphi_\infty(p).$$

By Lemma 4.1, for almost every $r_\infty > 0$, there is a subsequence (also denoted i) such that

$$(239) \quad d_{\mathcal{F}}(S(p_i, r_\infty), S(p, r_\infty)) \rightarrow 0.$$

Taking $r_\infty = \delta$, applying (231) we have

$$(240) \quad F_{i\#}S(p_i, r_\infty) = S(p'_i, r_\infty) \text{ where } p'_i = F_i(p_i)$$

so

$$(241) \quad d_{\mathcal{F}}(S(p'_i, r_\infty), S(p, r_\infty)) \rightarrow 0.$$

Combining via the triangle inequality with (237),

$$(242) \quad \liminf_{i \rightarrow \infty} d_{\mathcal{F}}(S(p'_i, r_\infty), \mathbf{0}) > 0.$$

Thus applying the basic Bolzano-Weierstrass Theorem [Theorem 7.1] to $S(p'_i, r_\infty)$, to see there is a $p_\infty \in \bar{X}'_\infty$ and a further subsequence (also denoted i) such that $p'_i \rightarrow p'_\infty$ in the sense that

$$(243) \quad \varphi'_i(p'_i) \rightarrow \varphi'_\infty(p'_\infty) \in Z'.$$

Define $F_\infty(p) = p_\infty$.

Repeat this process to choose subsequences and p_∞ for each p in the countable collection $P \subset X_\infty$. Diagonalize to obtain the subsequence in that statement of the theorem (also denoted M_i). Thus $F : P \rightarrow \bar{X}'_\infty$ is defined such that

$$(244) \quad \varphi_\infty(F_\infty(p)) = \lim_{i \rightarrow \infty} \varphi'_i(F_i(p_i)) \in Z'.$$

To see that F is distance preserving for any p, q in a ball of radius δ in X_∞ :

$$(245) \quad d_{\bar{X}'_\infty}(F_\infty(p), F_\infty(q)) = d_{Z'}(\varphi_\infty(F_\infty(p)), \varphi_\infty(F_\infty(q)))$$

$$(246) \quad = \lim_{i \rightarrow \infty} d_{Z'}(\varphi'_i(F_i(p_i)), \varphi'_i(F_i(q_i)))$$

$$(247) \quad = \lim_{i \rightarrow \infty} d_Z(\varphi_i(p_i), \varphi_i(q_i))$$

$$(248) \quad = d_Z(\varphi_\infty(p), \varphi_\infty(q)) = d_{X_\infty}(p, q).$$

In particular $F : P \rightarrow \bar{X}'_\infty$ is continuous and can be extended to the metric completion, $F_\infty : \bar{X}_\infty \rightarrow \bar{X}'_\infty$ which is an isometry on balls of radius δ .

For any sequence $q_i \in X_i$ converging to $q \in X_\infty$ one must show $F_i(q_i)$ converges to $F(q)$. Assume on the contrary that this fails:

$$(249) \quad \exists r_0 > 0 \exists N_0 \in \mathbb{N} \text{ s.t. } d_{Z'}(\varphi'_i(F_i(q_i)), \varphi'_i(F_\infty(q))) > r_0.$$

Since $q_i \rightarrow q$, there is an N_0 sufficiently large that

$$(250) \quad d_{X_\infty}(\varphi_i(q_i), \varphi_\infty(q)) \leq r_0/10. \quad \forall i \geq N_0.$$

Take $x_j \in P \subset X_\infty$ converging to q , and N_1 large enough that

$$(251) \quad d_Z(\varphi_\infty(x_j), \varphi_\infty(q)) < r_0/10 \quad \forall j \geq N_1.$$

For each i , take $x_{j,i} \in X_i$ converging to x_j such that $F_j(x_{j,i}) \rightarrow F_\infty(x_j)$. That is there exists N_i and N'_i sufficiently large that

$$(252) \quad d_Z(\varphi_i(x_{j,i}), \varphi_\infty(x_j)) < r_0/10 \quad \forall j \geq N_i$$

and

$$(253) \quad d_{Z'}(\varphi'_i(F_i(x_{j,i})), \varphi'_i(F_\infty(x_j))) < r_0/10 \quad \forall j \geq N'_i.$$

Since F_i and F_∞ are local isometries both are distance nonincreasing. In addition one has $F_i \circ \varphi_i = \varphi'_i \circ F_i$. Thus one has for $i \geq N_0$ and $j \geq \max\{N_1, N_i, N'_i\}$,

$$\begin{aligned} d_{Z'}(\varphi'_i(F_\infty(q)), \varphi'_i(F_i(q_i))) &\leq d_{Z'}(\varphi'_i(F_\infty(q)), \varphi'_i(F_\infty(x_j))) + d_{Z'}(\varphi'_i(F_\infty(x_j)), \varphi'_i(F_i(x_{j,i}))) \\ &\quad + d_{Z'}(\varphi'_i(F_i(x_{j,i})), \varphi'_i(F_i(q_i))) \\ &\leq d_{X'_\infty}(F_\infty(q), F_\infty(x_j)) + r_0/10 + d_{X'_i}(F_i(x_{j,i}), F_i(q_i)) \text{ by } j \geq N'_i, \\ &\leq d_{X_\infty}(q, x_j) + r_0/10 + d_{X_i}(x_{j,i}, q_i) \\ &\leq r_0/10 + r_0/10 + d_Z(\varphi_i(x_{j,i}), \varphi_i(q_i)) \text{ by } j \geq N_1, \\ &\leq r_0/5 + d_Z(\varphi_i(x_{j,i}), \varphi_\infty(x_j)) + d_Z(\varphi_\infty(x_j), \varphi_\infty(q)) + d_Z(\varphi_\infty(q), \varphi_i(q_i)) \\ &\leq r_0/5 + d_{X_i}(x_{j,i}, x_j) + d_{X_\infty}(x_j, q) + r_0/10 \text{ by } i \geq N_0, \\ &\leq r_0/5 + r_0/10 + r_0/10 + r_0/10 \text{ by } j \geq N_i, \end{aligned}$$

which contradicts (249).

To see that F_∞ is surjective when F_i are surjective, take any $x \in X'_\infty$. so

$$(254) \quad \liminf_{r \rightarrow 0} \mathbf{M}(S(x, r))/r^m > 0.$$

In particular

$$(255) \quad \exists r_x > 0 \text{ s.t. } S(x, r) \neq \mathbf{0} \quad \text{a.r. } r < r_x.$$

By Lemma 3.4 there exists $x_i \in X'_i$ such that

$$(256) \quad \lim_{i \rightarrow \infty} \varphi'_i(x_i) = \varphi'_\infty(x)$$

and by Lemma 4.1, for almost every $r > 0$ there is a subsequence (also denoted i) such that

$$(257) \quad d_{\mathcal{F}}(S(x_i, r), S(x, r)) \rightarrow 0.$$

Since F_i are surjective, there exists $p_i \in X_i$ such that $F_i(p_i) = x_i$. However, for almost every $r < \delta$,

$$(258) \quad F_{\#}S(p_i, r) = S(x_i, r)$$

so

$$(259) \quad d_{\mathcal{F}}(S(p_i, r), S(x, r)) \rightarrow 0.$$

and

$$(260) \quad \liminf_{i \rightarrow \infty} d_{\mathcal{F}}(S(p_i, r), \mathbf{0}) = h > 0.$$

Thus applying the basic Bolzano-Weierstrass Theorem [Theorem 7.1], there is a further subsequence of the p_i which converges to a $p_\infty \in X_\infty$. To see that $F_\infty(p_\infty) = x$ observe that

$$(261) \quad \varphi_\infty(F_\infty(p_\infty)) = \lim_{i \rightarrow \infty} \varphi_i(F_i(p_i))$$

$$(262) \quad = \lim_{i \rightarrow \infty} \varphi_i(x_i) = \varphi_\infty(x).$$

Now suppose $M_\infty = \mathbf{0}$. One needs only show that $M'_\infty = \mathbf{0}$. If not there exists $x \in X'_\infty$ such that (254)-(260) hold. However by Lemma 4.1

$$(263) \quad \lim_{i \rightarrow \infty} d_{\mathcal{F}}(S(p_i, r), \mathbf{0}) = 0$$

which contradicts (260). \square

Example 8.5. *The hypothesis that a uniform $\delta > 0$ exists such that (230) holds is necessary. This can be seen by taking M_i to be standard flat 1×1 tori and M'_i to be flat $1 \times (1/i)$ tori. Let $F_i : M_i \rightarrow M'_i$ be the i fold covering maps which are surjective local isometries on balls of radius $\delta_i = 1/(2i)$. Then M_i converges in the intrinsic flat sense to a standard flat torus while M'_i converges in the intrinsic flat sense to the $\mathbf{0}$ integral current space. Thus there cannot be any limit map F_∞ .*

9. EXAMPLE WITH NO INTRINSIC FLAT LIMIT

The theorems in this paper may be applied to prove certain sequences of Riemannian manifolds do not converge or converge to specific Riemannian manifolds. One such example is provided here. Further examples will appear in joint work with Basilio [BS14].

Example 9.1. *There exists a sequence of smooth Riemannian manifolds with boundary with constant sectional curvature such that $\text{Vol}_{m-1}(\partial M_j) \leq A_0$, $\text{Diam}(M_j) \leq D_0$ such that no subsequence converges in the intrinsic flat or Gromov-Hausdorff sense not even to $\mathbf{0}$.*

Proof. Let M_j be the j fold covering space of

$$(264) \quad N_j = \mathbb{S}^2 \setminus (B_{p_+}(1/j), B_{p_-}(1/j))$$

where \mathbb{S}^2 is endowed with the standard metric tensor $g_{\mathbb{S}^2}$ which is lifted to M_j and p_+ and p_- are opposite poles. Let d_j be the length metric on M_j defined by this metric tensor.

Then

$$(265) \quad \text{Diam}(M_j) \leq \pi + j2\pi(1/j) + \pi = 4\pi$$

and

$$(266) \quad \text{Vol}_{m-1}(\partial M_j) \leq j \text{Vol}_{m-1}(\partial N_j) \leq j2(2\pi/j) = 4\pi$$

but

$$(267) \quad \lim_{j \rightarrow \infty} \frac{\text{Vol}_m(M_j)}{j} = \lim_{j \rightarrow \infty} \text{Vol}(N_j) = \text{Vol}(\mathbb{S}^2) = 4\pi.$$

Suppose on the contrary that a subsequence converges $M_{j'} \xrightarrow{\mathcal{F}} M_\infty$.

Case I: $M_\infty = \mathbf{0}$. If so, then by Lemma 4.1, any sequence $q_j \in M_j$ and almost every $r > 0$, there is a subsequence $S(q_{j'}, r) \xrightarrow{\mathcal{F}} \mathbf{0}$. Take q_j lying on the equator and choose an $r < 1/2$. Then by the convexity of balls one has

$$(268) \quad S(q_j, r) = \left(B(p_0, r), d_{\mathbb{S}^2}, \int_{B(p_0, r)} \right)$$

are all isometric to one another. Thus they do not converge to $\mathbf{0}$ and there is a contradiction.

Case II: $M_\infty \neq \mathbf{0}$. Let $x_{j,1}, x_{j,2}, \dots, x_{j,j}$ lie on the equator of X_j so that

$$(269) \quad d_{X_j}(x_{j,i}, x_{j,k}) \geq \pi \quad \forall i, k \in \{1, 2, \dots, j\}.$$

Observe also that $B(x_{j,k}, \pi/4)$ are disjoint and are all isometric to a ball $B(x, \pi/4)$ in a standard sphere. Thus

$$(270) \quad d_{\mathcal{F}}(S(x_{j,k}, \pi/4), S(x, \pi/4)) = 0 \quad \forall k \in \{1, 2, \dots, j\}.$$

and

$$(271) \quad d_{\mathcal{F}}(S(x_{j,k}, \pi/4), \mathbf{0}) = h_0 = d_{\mathcal{F}}(S(x, \pi/4), \mathbf{0}) > 0 \quad \forall k \in \{1, 2, \dots, j\}.$$

Applying Theorem 7.1, there is a subsequence of each $x_{k,j}$ must converge to some $x_k \in \bar{X}_\infty$. Diagonalizing, there is a subsequence (also denoted M_j) such that $x_{k,j} \rightarrow x_k$ for all k : so that

$$(272) \quad d_{X_\infty}(x_k, x_{k'}) \geq \pi$$

so that $B(x_{j,k}, \pi/4)$ are disjoint. Applying Lemma 4.1,

$$(273) \quad \lim_{j \rightarrow \infty} d_{\mathcal{F}}(S(x_{j,k}, \pi/4), S(x_k, \pi/4)) = 0.$$

and so

$$(274) \quad d_{\mathcal{F}}(S(x_k, \pi/4), S(x, \pi/4)) = 0.$$

Thus M_∞ contains infinitely many balls of the same mass, which contradicts the fact that $\mathbf{M}(T_\infty)$ is finite. \square

10. POSSIBLE FURTHER APPLICATIONS

In this section a number of possible applications of the theorems in this paper are discussed.

Remark 10.1. In [BI95], Burago and Ivanov prove that the volume growth of the universal cover of a Riemannian manifold homeomorphic to a torus is at least that of Euclidean space. If it is exactly equal, then they have a rigidity theorem stating that the Riemannian manifold is flat. Theorem 8.1 may be useful in the study of questions arising in Gromov's work [Gro12] analyzing the almost rigidity of Burago-Ivanov's Theorem (where the volume growth is close to that of Euclidean space).

Remark 10.2. *Theorem 8.1 should be useful when wishing to study limits of covering maps and analyzing the existence of a universal cover of an intrinsic flat limit. Recall that in joint work with Wei, the author has conducted such an analysis of Gromov-Hausdorff limits [SW01].*

Remark 10.3. *Theorem 8.1 should also be useful when studying how covering spectra behave under intrinsic flat convergence. See joint work of the author with Wei in which it was shown that covering spectra behave continuously under Gromov-Hausdorff convergence [SW04a].*

Remark 10.4. *Theorem 6.1 may possibly be applied to study the limits of harmonic functions, eigenfunctions and heat kernels. Recall that Cheeger-Colding proved the convergence of eigenfunctions and eigenvalues when the Riemannian manifolds are converging in the measured Gromov-Hausdorff sense with a uniform lower bound on Ricci curvature [CC00]. Ding has proved the convergence of heat kernels in this setting [Din02]. Building on work of Fukaya [Fuk87], Sinaei has proven the convergence of harmonic maps in this setting with additional conditions [Sin13]. Portegies has shown that eigenvalues need not converge when one only has intrinsic flat convergence without a volume bound, but building on work of Fukaya [Fuk87] has shown the eigenvalues semiconverge as long as the volume converges [Por14]. It would be interesting to examine what happens to the eigenfunctions and heat kernel in this setting.*

Remark 10.5. *Recall that in [SW10], the author and Wenger proved that intrinsic flat and Gromov-Hausdorff limits agree when the sequence of manifolds has nonnegative Ricci curvature and no boundary. In particular, there are no disappearing sequences of points in this setting. Theorems 6.1 and 7.1 may possibly be combined with theorems of Cheeger-Colding in [CC97], to prove that there are no disappearing points as long as the manifolds have a uniform lower bound on Ricci curvature.*

If a reader is interested in studying any of these questions, please contact the author. More details can be provided and the author can coordinate the research of those working on these problems. Funding to visit the author may be available.

REFERENCES

- [AK00] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.
- [BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
- [BI95] D. Burago and S. Ivanov. On asymptotic volume of tori. *Geom. Funct. Anal.*, 5(5):800–808, 1995.
- [BS14] Jorge Babilio and Christina Sormani. Sequences of three-dimensional manifolds with positive scalar curvature. *preprint to appear*, 2014.
- [CC97] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46(3):406–480, 1997.
- [CC00] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. III. *J. Differential Geom.*, 54(1):37–74, 2000.
- [CN11] J Cheeger and A Naber. Lower bounds on Ricci curvature and quantitative behavior of singular sets. *arXiv:1103.1819*, 2011.
- [Din02] Yu Ding. Heat kernels and Green’s functions on limit spaces. *Comm. Anal. Geom.*, 10(3):475–514, 2002.
- [FF60] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.
- [Fuk87] Kenji Fukaya. Collapsing of Riemannian manifolds and eigenvalues of Laplace operator. *Invent. Math.*, 87(3):517–547, 1987.
- [GP91] Karsten Grove and Peter Petersen. Manifolds near the boundary of existence. *J. Differential Geom.*, 33(2):379–394, 1991.

- [Gro81] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.
- [Gro99] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.
- [Gro12] Misha Gromov. Hilbert volume in metric spaces. Part 1. *Cent. Eur. J. Math.*, 10(2):371–400, 2012.
- [LS13] Sajjad Lakzian and Christina Sormani. Smooth convergence away from singular sets. *Comm. Anal. Geom.*, 21(1):39–104, 2013.
- [LS14] Dan A. Lee and Christina Sormani. Stability of the positive mass theorem for rotationally symmetric riemannian manifolds. *Journal für die Reine und Angewandte Mathematik (Crelle's Journal)*, 686, 2014.
- [Per03] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv:math/0303109*, 2003.
- [Por14] Jacobus Portegies. Semicontinuity of eigenvalues under intrinsic flat convergence. *arxiv:1401.5017*, 2014.
- [Rud87] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
- [Sin13] Zahra Sinaei. Convergence of harmonic maps. *arXiv:1209.5893*, 2013.
- [Sor04] Christina Sormani. Friedmann cosmology and almost isotropy. *Geom. Funct. Anal.*, 14(4):853–912, 2004.
- [Sor13] Christina Sormani. Properties of the intrinsic flat distance. *arXiv:1210.3895*, 2013.
- [SW01] Christina Sormani and Guofang Wei. Hausdorff convergence and universal covers. *Trans. Amer. Math. Soc.*, 353(9):3585–3602 (electronic), 2001.
- [SW04a] Christina Sormani and Guofang Wei. The covering spectrum of a compact length space. *J. Differential Geom.*, 67(1):35–77, 2004.
- [SW04b] Christina Sormani and Guofang Wei. Universal covers for Hausdorff limits of noncompact spaces. *Trans. Amer. Math. Soc.*, 356(3):1233–1270 (electronic), 2004.
- [SW10] Christina Sormani and Stefan Wenger. Weak convergence and cancellation, appendix by Raanan Schul and Stefan Wenger. *Calculus of Variations and Partial Differential Equations*, 38(1-2), 2010.
- [SW11] Christina Sormani and Stefan Wenger. Intrinsic flat convergence of manifolds and other integral current spaces. *Journal of Differential Geometry*, 87, 2011.
- [Wen07] Stefan Wenger. Flat convergence for integral currents in metric spaces. *Calc. Var. Partial Differential Equations*, 28(2):139–160, 2007.
- [Whi57] Hassler Whitney. *Geometric integration theory*. Princeton University Press, Princeton, N. J., 1957.

CUNY GRADUATE CENTER AND LEHMAN COLLEGE

E-mail address: sormanic@member.ams.org